2(a). Waves and optics

Part – A

1. **Define simple harmonic motion?**
   If the motion is periodic such that acceleration is directly proportional to displacement and is always directed towards its mean (equilibrium) position
2. **Define amplitude and phase?**
   The maximum distance covered by the body on either side of its mean position is called amplitude.
   Phase is a physics quantity that expresses the instantaneous position and direction of motion of an oscillating system.
3. **Define Time period and frequency, Give its relation?**
   The smallest time required to complete one vibration (or) oscillation is known as time period.
   Time period $T = 2\pi\sqrt{\frac{Displacement}{Acceleration}}$
   The number of oscillation made by a body per second is known as frequency of oscillation. it is the reciprocal of the time period
4. **What is meant by free oscillation?**
   A body which vibrates freely with its natural frequency are said to be free oscillation
5. **What is meant by damped oscillation?**
   If a body is set in to vibration, the amplitude keeps on decreasing because of the frictional resistance to the motion and hence after some time the oscillation drops down to zero. This type of oscillation is called damped oscillation.
   Eg: when a pendulum is displaced from the equilibrium position, it oscillates with decreasing amplitude and finally comes to rest.
6. **What is meant by forced vibration?**
   In this type of oscillation, we need to give an external force for the oscillation to sustain. Hence the body vibrates with a frequency other than natural frequency in equal interval of time.
   Eg: Floor vibrates due to marching of soldiers.
7. **What do you understand by the term DEAD BEAT? Give example?**
   When the oscillator is under motion, the displacement suddenly drops down to zero without performing any oscillation. Such motions are set to be over damped or dead beat oscillations
   Eg: Dead beat moving coil galvanometer
8. **What is meant by CRITICAL DAMPING? give example**
   During oscillatory motion, when the displacement decreases to zero rapidly, then it is called critical damped motion.
   Eg: Movement of pointers in ammeter, voltmeters, etc.,
9. **What do you infer from the amplitude and phase in forced oscillation?**

we know that $A = \dfrac{f}{\sqrt{\left[\left(\omega_0^{\,2} - \omega^2\right)^2 + 4b^2\omega^2\right]}}$ and phase $\theta = \tan^{-1}\left[\dfrac{2b\omega}{\omega_0^{\,2} - \omega^2}\right]$

From these equations, it is clear that the amplitude and phase of forced oscillation depend on driving frequency ω and natural frequency ω_0 of the oscillator.

10. What is meant by resonance? Give examples.

When the driving frequency p matches with natural frequency ω, resonance occurs.

Eg: collapse of bridges and roads due to earthquake.

11. Define progressive wave?

It is defined as the vibratory motion of a body which is transmitted continuously in the same direction from one particle to the successive particle of the medium and travel forward through the medium due to its elastic property.

12. Define longitudinal wave?

It is a wave motion in which the particles of the medium are vibrate about their mean position along the direction of propagation of wave

13. Define transverse wave?

It is a wave motion in which the particles of the medium are vibrate about their mean position perpendicular to the direction of propagation of wave

14. Define plane progressive wave?

A plane progressive wave is the simplest wave in which the particle of the medium perform simple harmonic motion.

15. Define relaxation time for damped oscillator?

It is defined as the time taken for the total mechanical energy to decay (1/e) of its original value. i.e., $E = E_0 e^{-\frac{t}{\tau}}$

16. Define logarithmic decrement of damped oscillator?

It is defined as the natural logarithm of the ratio between two successive maximum amplitudes which are separated by time period

17. Define quality factor?

It is defined as 2π times the ratio of energy stored in the system to the energy lost per period

$$Q = 2\pi \frac{energy stored}{energy lost} = 2\pi \frac{E}{\frac{E}{\tau} \times T} = \omega\tau$$

Part – B

1. **Develop the general theory of the damped oscillation and discuss its essential cases involved in it?**

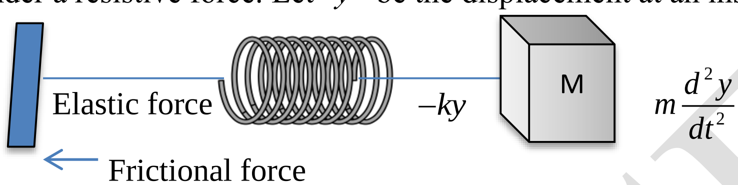
Damped oscillation:

If a body is set in to vibration, the amplitude keeps on decreasing because of the frictional resistance to the motion and hence after some time, the oscillation drops down to zero. This type of oscillation is called damped oscillation.

Eg: when a pendulum is displaced from the equilibrium position, it oscillates with decreasing amplitude and finally comes to rest.

Differential equation and its solution

Let us consider a body of mass "*m*" attaching to a spring executing simple harmonic motion under a resistive force. Let "*y*" be the displacement at an instant of time "*t*" .



Then damping system is subjected to:

(i) **Restoring force** which is proportional to the displacement and it is acting in opposite direction. i.e., $F \infty y$ (or) $F = -ky$ (1)

(ii) **Frictional force** which is proportional to velocity and directed in the opposite direction of motion i.e., $F \infty \frac{dy}{dt}$ (or) $F = -r\frac{dy}{dt}$ (2)

Then, Total instantaneous force acting on the body is $F = -ky - r\frac{dy}{dt}$ (3)

But, from Newton's II law of motion, Resultant force (F) = mass (m) $\times$ acceleration ($\frac{d^2y}{dt^2}$)

Hence equation (3) becomes, $m\frac{d^2y}{dt^2} = -ky - r\frac{dy}{dt}$

(or) $\frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{k}{m}y = 0$

(or) $\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = 0$ (4)

Where $2b = \frac{r}{m}$ and $\omega^2 = \frac{k}{m}$. Also b is damping factor (or) damping coefficient.

Equation (4) is a second order differential equation of **Damped harmonic motion.**

The solution of equation (4) is $y = Ae^{\alpha t}$ (5)

Where A and α are constant and can be determined from boundary conditions.

Differentiating equation (5) two times with respect to t,

$\frac{dy}{dt} = \alpha Ae^{\alpha t}$ and (6)

$$\frac{d^2y}{dt^2} = \alpha^2 A e^{\alpha t} \quad (7)$$

Substituting the values of equations 5, 6, and 7 in equation 4, we get

$$A\alpha^2 e^{\alpha t} + 2bA\alpha e^{\alpha t} + \omega^2 A e^{\alpha t} = 0$$

(or) $Ae^{\alpha t}\ (\alpha^2 + 2b\alpha + \omega^2) = 0$

As $Ae^{\alpha t} \neq 0, \therefore \alpha^2 + 2b\alpha + \omega^2 = 0$ (8)

The solution of equation (8) is

$$\alpha = \frac{-2b \pm \sqrt{4b^2 - 4\omega^2}}{2}$$

(or) $\alpha = -b \pm \sqrt{b^2 - \omega^2}$ (9)

Hence, the general solution of equation (5) is

$$y = A_1 e^{[-b+\sqrt{b^2-\omega^2}]t} + A_2 e^{[-b-\sqrt{b^2-\omega^2}]t} \quad (10)$$

where A_1 and A_2 are arbitrary constant, whose values are determined from the boundary conditions.

Case : 1 Heavy damping

when $b^2 >> \omega^2$, then $\sqrt{b^2 - \omega^2}$ is real and less than *b*, hence the powers in equation (9) are negative. Thus the displacement exponentially reduces to zero without performing any oscillation. This type of motion is known as **over damped or dead beat.**

Eg: Dead beat moving coil galvanometer

Case : 2 Critical damping

when $b^2 = \omega^2$, then $\sqrt{b^2 - \omega^2} = h \to 0$ i.e., the value of *h* is very close to zero.

Hence equation (10) reduces to

$$y = A_1 e^{[-b+h]t} + A_2 e^{[-b-h]t}$$

(or) $y = A_1 e^{-bt} e^{ht} + A_2 e^{-bt} e^{-ht}$

(or) $y = e^{-bt}\left[A_1 e^{ht} + A_2 e^{-ht}\right]$

(or) $y = e^{-bt}\left[A_1(1 + ht + ..) + A_2(1 - ht + ...)\right]$

(or) $y = e^{-bt}\left[(A_1 + A_2) + h(A_1 - A_2)t\right]$

(or) $y = e^{-bt}\left[p + qt\right]$ (11)

where $p = (A_1 + A_2)$ and $q = h(A_1 - A_2)$

From equation (11), as displacement increases, the term $\left[p + qt\right]$ increases, but due to negative exponential term, the displacement rapidly reduces to zero and such a motion is called **critical damping**

Eg: Pointer instruments such as voltmeter, ammeter, etc.,

Case : 2 Under damping

when $b^2 << \omega^2$, then $\sqrt{b^2-\omega^2}$ is negative and imaginary .

i.e., $\sqrt{b^2-\omega^2} = i\sqrt{\omega^2-b^2} = i\beta$

where $\beta = \sqrt{\omega^2-b^2}$ and $i = \sqrt{-1}$

equation (9) now becomes, $y = A_1 e^{[-b+i\beta]t} + A_2 e^{[-b-i\beta]t}$ (12)

(or) $y = A_1 e^{-bt} e^{i\beta t} + A_2 e^{-bt} e^{-i\beta t}$

(or) $y = e^{-bt}[A_1 e^{i\beta t} + A_2 e^{-i\beta t}]$

(or) $y = e^{-bt}[A_1(\cos\beta t + i\sin\beta t) + A_2(\cos\beta t - i\sin\beta t)]$

(or) $y = e^{-bt}[(A_1 + A_2)\cos\beta t + (A_1 - A_2)i\sin\beta t)]$

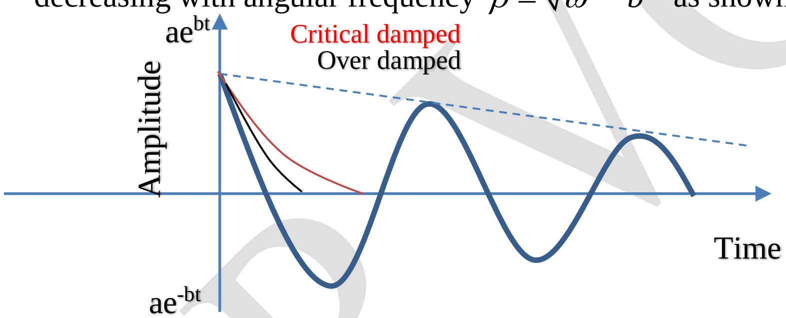
$y = e^{-bt}[a\sin\phi\cos\beta t + a\cos\phi\sin\beta t)]$

where $a\sin\phi = (A_1 + A_2)$ and $a\cos\phi = i(A_1 - A_2)$

$\therefore y = ae^{-bt}\sin\left(\beta t + \phi\right)$

(or) $\therefore y = ae^{-bt}\sin\left((\sqrt{\omega^2-b^2})t + \phi\right)$ (13)

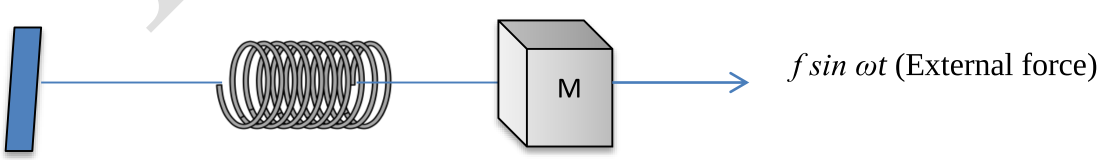
this is the equation of a under damped harmonic motion with amplitude ae^{-bt} and goes on decreasing with angular frequency $\beta = \sqrt{\omega^2-b^2}$ as shown in curve.



2. **Develop the general theory of the forced oscillation and discuss its essential cases involved in it?**

In this type of oscillation, an external force is given for the oscillation to sustain. Hence the body vibrates with a frequency other than natural frequency due to external force applied in equal interval of time.

Eg: Floor vibrating due to marching of soldiers.



Differential equation and its solution

Let us consider a body of mass "*m*" attaching to a spring and an external force is applied in order to sustain the simple harmonic motion

Here three types of forces acting on this oscillator

(i)**Restoring force** which is proportional to the displacement and it is acting in opposite direction. i.e., $F \infty y$ (or) $F = -ky$ (1)

(ii)**Frictional force** which is proportional to velocity and directed in the opposite direction of motion i.e., $F \infty \frac{dy}{dt}$ (or) $F = -r\frac{dy}{dt}$ (2)

(iii)**External force** which is opposite to the above two forces and helps in maintaining the oscillation given by $F \sin pt$ (3)

where F is the maximum external force and p is the driving frequency of the forced oscillator

Then, Total instantaneous force acting on the body is $F = -ky - r\frac{dy}{dt} + F\sin\omega t$ (4)

But, from Newton's II law of motion, Resultant force (F) = mass (m) $\times$ acceleration ($\frac{d^2y}{dt^2}$)

Hence equation (3) becomes, $m\frac{d^2y}{dt^2} = -ky - r\frac{dy}{dt} + F\sin\omega t$

(or) $\frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{F}{m}\sin\omega t$

(or) $\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = f\sin\omega t$ (5)

where $2b = \frac{r}{m}$; $\omega_0{}^2 = \frac{k}{m}$ and $f = \frac{F}{m}$. Also b is damping factor (or) damping coefficient.

Equation (5) is a second order differential equation of **Forced harmonic motion.**

The solution of equation (5) is $y = A\sin(\omega t - \theta)$ (6)

where A is the steady amplitude of vibration and θ is the angle at which the displacement *'y'* lag behind the applied force $f\sin\omega t$

Differentiating equation (5) twice with respect to time, we get

$$\frac{dy}{dt} = A\omega\cos(\omega t - \theta) \quad (7)$$

$$\frac{d^2y}{dt^2} = -A\omega^2\sin(\omega t - \theta) \quad (8)$$

Substituting equations (6), (7) & (8) in (5), we get

$$-A\omega^2\sin(\omega t - \theta) + A2b\omega\cos(\omega t - \theta) + \omega_0{}^2\sin(\omega t - \theta) = f\sin\left[\left(\omega t - \theta\right) + \theta\right]$$

(or) $A(\omega_0{}^2 - \omega^2)\sin(\omega t - \theta) + 2bA\omega\cos(\omega t - \theta) = f\sin(\omega t - \theta)\cos\theta + f\cos(\omega t - \theta)\sin\theta$ (9)

Equation (9) holds good for all the values of t. Hence the coefficients of $\sin(pt - \theta)$ and $\cos(\omega t - \theta)$ must be equal on both sides

$$\therefore A\left(\omega_0{}^2 - \omega^2\right) = f\cos\theta \quad (9)$$

Similarly $2b\omega A = f\sin\theta$ (10)

Squaring and adding equation (9) & (10)

$$A^2\left(\omega_0{}^2-\omega^2\right)^2+4b^2\omega^2A^2 = f^2\cos^2\theta + f^2\sin^2\theta$$

(or) $A^2\left(\omega_0{}^2-\omega^2\right)^2+4b^2\omega^2A^2 = f^2\left[\because \cos^2\theta+\sin^2\theta=1\right]$

(or) $A^2\left[\left(\omega_0{}^2-\omega^2\right)^2+4b^2\,\omega^2\right]= f^2$

(or) $A^2=\dfrac{f^2}{\left(\omega_0{}^2-\omega^2\right)^2+4b^2\omega^2}$

(or) $A=\omega\dfrac{f}{\sqrt{\left(\omega_0{}^2-\omega^2\right)^2+4b^2\omega^2}}$ (11)

Dividing equation (10) by (9)

$$\frac{2b\omega A}{A(\omega_0{}^2-\omega^2)}=\frac{f\sin\theta}{f\cos\theta}$$

(or) $\tan\theta=\dfrac{2b\omega A}{A(\omega_0{}^2-\omega^2)}$

(or) $\theta=\tan^{-1}\left[\dfrac{2b\omega A}{A(\omega_0{}^2-\omega^2)}\right]$ (12)

From equations (11) & (12), it is clear that the amplitude and phase of forced oscillation depends on driving frequency (ω) and natural frequency (ω_0) of the oscillator.

Special cases

Case: 1 when driving frequency (ω) is less than natural frequency (ω_0). i.e., $\omega_0{}^2>\omega^2$

Amplitude $A=\dfrac{f}{\sqrt{\left(\omega_0{}^2-\omega^2\right)^2+4b^2\omega^2}}\cong\dfrac{f}{\omega_0{}^2}$

since $f=\dfrac{F}{m}$, then $A=\dfrac{F}{m\omega_0{}^2}=\dfrac{Fm}{km}=\dfrac{F}{k}$

Hence amplitude depends on force constant k and magnitude of applied force

Phase

$$\theta=\tan^{-1}\left[\frac{2b\omega A}{A(\omega_0{}^2-\omega^2)}\right]$$

since $\omega^2\gg p^2$, hence the term $\dfrac{2b\omega}{\omega_0}\to 0$

& hence $\theta=\tan^{-1}\left[0\right]=0$

Hence under this situation, the displacement and driving force are in phase with each other

Case: 2 when driving frequency (p) is equal to natural frequency (ω). 1.e., $\omega^2 = p^2$, this condition is said to be **resonance**

Amplitude $A = \dfrac{f}{\sqrt{\left(\omega_0{}^2 - \omega^2\right)^2 + 4b^2\omega^2}} = \dfrac{f}{2b\omega}$

since $f = \dfrac{F}{m}$ and $2b = \dfrac{r}{m}$, then $A = \dfrac{Fm}{mr\omega} = \dfrac{F}{r\omega}$

Hence amplitude depends on damping force (r) and magnitude of applied force

Phase

since $\omega_0{}^2 = \omega^2$, hence $\theta = \tan^{-1}\left[\dfrac{2b\omega}{0}\right]$

& hence $\theta = \tan^{-1}\left[\alpha\right] = \dfrac{\pi}{2}$

Hence under this situation, the displacement lags behind the driving force by a phase of $\dfrac{\pi}{2}$

Case: 3 when driving frequency (ω) is greater than natural frequency (ω_0). i.e., $\omega_0{}^2 < \omega^2$

Amplitude $A = \dfrac{f}{\sqrt{\left(\omega_0{}^2 - \omega^2\right)^2 + 4b^2\omega^2}} \cong \dfrac{f}{\omega^2}$

since $f = \dfrac{F}{m}$, then $A = \dfrac{F}{m\omega^2}$

Hence amplitude depends on applied force and mass of the body

Phase

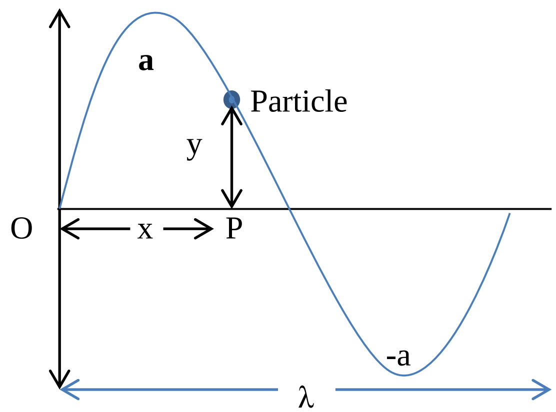
$\theta = \tan^{-1}\left[\dfrac{2b\omega}{(-\omega^2)}\right]$ (or) $\theta = \tan^{-1}\left[\dfrac{2b}{(-\omega)}\right]$

since ω is very large, hence the term $\dfrac{1}{\omega} = 0$

& hence $\theta = \tan^{-1}\left[-0\right] = \pi$

Hence under this situation, the displacement lags behind the driving force by 180°

3. Derive an expression for the particle velocity for a plane progressive wave and obtain the differential equation for the moving along positive X – direction



A plane progressive wave is the simplest wave in which the particle of the medium perform simple harmonic motion

Explanation

Let us consider a plane progressive simple harmonic wave, originating from the origin "O" and travels towards +ve X direction. As the wave propagates, each successive particle of the medium is set into simple harmonic motion.

Displacement at O

If "*v*" is the velocity of the particle and "*y*" is the displacement of the particle at any time "*t*" then we can write displacement of the particle at "O" is $y = A\sin\omega t$ (1)

where *A* is the amplitude, we know that $\omega = \dfrac{2\pi}{T}$ where *T* is the time period of oscillation. The time taken by a wave to cover a distance "λ" is the wavelength

Hence $y = A\sin\dfrac{2\pi}{T}t$ (2)

Displacement at P

Now, consider a particle *P* at a distance *x* from *O*. here the wave starting from *O* will reach *P* after (*t* = *x/v*) seconds. i.e., the particle P will have a time delay of $\dfrac{x}{v}$ seconds from the particle *O* as $\left(t - \dfrac{x}{v}\right)$ seconds we can write displacement of the particle at "P" is $y = A\sin\frac{2\pi}{T}\left[t-\frac{x}{v}\right]$ (3)

$\therefore y = A\sin 2\pi\left[\frac{t}{T}-\frac{x}{Tv}\right]$

we know that $\lambda = Tv(or)T = \frac{\lambda}{v}$

hence equation (3) becomes, $y = A\sin\frac{2\pi}{\lambda}(vt-x)$ (4)

This equation represents the complete form of plane progressive wave propagating with the velocity "v" in positive X – direction. similarly for negative X – direction, the plane progressive wave equation is $y = A\sin\frac{2\pi}{\lambda}(vt+x)$ (5)

Particle velocity

Particle velocity is defined as the rate of change of displacement "y" with respect to time "t"

Differentiating equation (5) two times with respect to t

$\frac{dy}{dt} = A\frac{2\pi v}{\lambda}\cos\frac{2\pi}{\lambda}(vt-x)$ and (6)

$\frac{d^2y}{dt^2} = -\frac{4\pi^2v^2}{\lambda^2}A\sin\frac{2\pi}{\lambda}(vt-x)$ (7)

Compressible is due to change in +ve direction with respect to distance "x" and rarefaction is due to change in –ve direction with respect to distance "x". Hence differentiating equation (5) two times with respect to "x"

$\frac{dy}{dx} = -\frac{2\pi A}{\lambda}\cos\frac{2\pi}{\lambda}(vt-x)$ (8)

$\frac{d^2y}{dx^2} = -\frac{4\pi^2}{\lambda^2}A\sin\frac{2\pi}{\lambda}(vt-x)$ (9)

comparing equation (6) & (8)

Particle velocity $\left(\frac{dv}{dt}\right)$ = - wave velocity (V) x slope of displacement $\left(\frac{dy}{dx}\right)$ (10)

comparing equation (7) & (9)

$$\frac{d^2y}{dt^2} = \frac{1}{v^2}\frac{d^2y}{dx^2} \tag{11}$$

This equation is the differential equation of a plane progressive wave .