1. Mechanics

Objective:

- **To make the students effectively to achieve an understanding of mechanics.**

Syllabus:

Multiparticle dynamics: Center of mass (CM) - CM of continuous bodies - motion of the CM - kinetic energy of system of particles. Rotation of rigid bodies: Rotational kinematics - rotational kinetic energy and moment of inertia - theorems of M .I - moment of inertia of continuous bodies - M.I of a diatomic molecule - torque - rotational dynamics of rigid bodies - conservation of angular momentum - rotational energy state of a rigid diatomic molecule - gyroscope - torsional pendulum - double pendulum - Introduction to nonlinear oscillations.

1. Introduction

Mechanics is a branch of physics which deals with the motion of bodies under the action of forces. In elementary mechanics, most of the bodies are assumed to be rigid. But in actual practice, no body is perfectly rigid. When a stationery body is acted upon by some external forces, then the body may start to rotate (or) move about any point. If the body doesn't move (or) rotate then it is said to be in equilibrium.

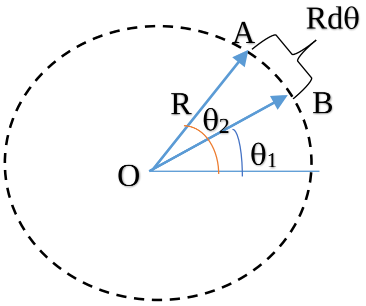
We know that the rigid body is the combination of many particles i.e., multiparticle. Let us discuss the basic definitions relate to mechanics

(1) Angular displacement

Definition

The change in position of the particle moving in a circular path with respect to an angle (dθ) is called angular displacement.

Proof



Let us consider a particle of mass *m* moving in a circular path of radius 'R' with respect to the center of the circle O. At *t=0 sec*, the particle is located at the point *A* and after time interval *t* , it reaches the point *B* as shown in figure.

W.K.T., the angular displacement of a particle is the change in angular Position between two points A and B, which can be measured by the angle $(\theta_2-\theta_1)$ between the radius vector of these two positions *A* and *B*.

∴ the angle between *A* and *B* is $d\theta = (\theta_2 - \theta_1)$.

Angular displacement $d\theta = (\theta_2 - \theta_1)$. (unit: Radian)

We can write arc length as $AB = l$,

Then the relation between angular displacement (dθ) and linear displacement (l) is given by its arc length as $l = R\, d\theta$.

(2) Angular velocity

The rate of change of angular displacement is called angular velocity

i.e., Angular velocity (ω) = $d\theta / dt$.   (unit : Rad s^{-1})

The relation between angular velocity (ω) and linear velocity (v) is given by $v = r\,\omega$.

(3) Angular acceleration

The rate of change of angular velocity is called angular acceleration.

i.e., Angular acceleration (α) = $d\omega/dt$ (or) $d^2\theta / dt^2$. (Unit: Rad s^{-2}).

(4) Angular momentum

The moment of inertia times of angular velocity of the particle is called angular momentum.

i.e., Angular momentum $L = I\,\omega$ (Unit : kgm^2s^{-1})

(5) Inertia

It is the tendency of an object to maintain its state of rest or of uniform motion along the same direction. Inertia is a resisting capacity of an object to alter its state of rest and motion (direction and /or magnitude).

1.1. Multiparticle dynamics (Dynamics in a system of particles)

We know dynamics is the study of motion of bodies under the action of forces. Multiparticle dynamics (dynamics in a system of particles) is the study of motion in respect of a group of particles in which the separation between the particles will be very small i.e., the distance between the particles will be negligible.

Explanation

In dynamics, we study the physical parameters by considering an object as a point mass and its shape and size is ignored. But, in real world problems, object will execute rotational and translational motion. For example, if we kick the football, it has both translational and rotational motions. As both the motion depends on the size and shape of the object, both cannot be ignored, even it is negligible. Thus, the study of rotational and translator motion with respect to the system of particles is called multi-particle dynamics.

1.2. Centre of mass

We know that mass is the measure of the body’s resistance to change the motion (or) it is measure of inertia of the body. It is a scalar quantity and it is constant.

(i) A system consists of many particles with different masses and different position from the reference point.

(ii) The mass of the system is equal to the sum of the mass of each particle in the system.

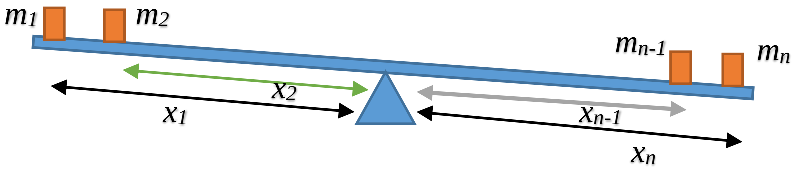
Hence, if the mass of the entire particles of the system is concentrated at a particular point, that point is called centre of mass of the system.

1.3. Centre of mass in a one dimensional system

The system consists of many particles with different positions and different masses. If the mass of the entire particle in the system is concentrated at a particular point, then that point is called centre of mass of the system.

Explanation

Let us consider a fulcrum placed along the x axis which is not at equilibrium position as shown in figure.



Let the position of masses m_1, m_2, m_3, ….., m_{n-1}, m_n be at a distance of x_1,x_2,……..,x_{n-1},x_n respectively from the fulcrum. *The tendency of a mass to rotate with respect to origin or supporting point is called moment of mass.*

The moment of mass for an elemental mass m_n with respect to the fulcrum can be written as m_nx_n. If the moments on both sides are equal, then the system is sai to be in equilibrium. Therefore, total moments with respect to the fulcrum shall be written as

$$m_1x_1 + m_2x_2 + ... + m_nx_n = \sum_{i=1}^{N} m_ix_i = 0 \qquad (1)$$

If the total moment is equal to zero, then the centre of mass will lie at the supporting point (or) fulcrum and the system is said to be in equilibrium. If the fulcrum is placed at the unbalanced position, then it is shifted to a balanced position (say of distance X) to reach the equilibrium position.

Under equilibrium condition,

$$\sum_{i=1}^{n} m_ix_i - \sum_{i=1}^{n} m_iX = 0$$

$$\text{(or)} \quad \sum_{i=1}^{n} m_ix_i = \sum_{i=1}^{n} m_iX$$

$$\text{(or)} \quad \sum_{i=1}^{n} m_ix_i = X\sum_{i=1}^{n} m_i$$

$$\text{(or)} \quad X = \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i} \qquad (2)$$

Where $\sum_{i=1}^{n} m_i x_i$ is the moment of system and

$\sum_{i=1}^{n} m_i$ is the mass of the system

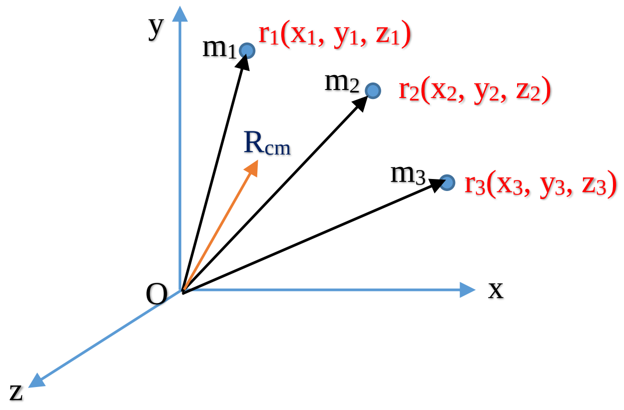
Thus, the system should be move to a distance of X metres in order to attain the balanced position of the system.

The distanced moved to obtain equilibrium position (or) so called the centre of mass in a one dimensional system is given by

$$X = \frac{m_1 x_1 + m_2 x_2 + ...}{m_1 + m_2 + ...} \qquad (3)$$

1.4. Centre of mass in three dimensional system

To find the centre of mass in a three dimensional system, let us consider a three dimensional system in which let m_1, m_2, m_3,…. be the masses placed at position vectors $r_1(x_1,y_1, z_1)$, $r_2(x_2,y_2,z_2)$,…. Respectively from the origin 'O' as shown in figure



Here,

(i) The centre of mass along the x – axis,

$$X = \frac{m_1 x_1 + m_2 x_2 + ...}{m_1 + m_2 + ...} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

(ii) The centre of mass along y-axis,

$$Y = \frac{m_1 y_1 + m_2 y_2 + ...}{m_1 + m_2 + ...} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}$$

(iii) The centre of mass along z-axis,

$$Z = \frac{m_1 z_1 + m_2 z_2 + \ldots.}{m_1 + m_2 + \ldots..} = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i}$$

In general, centre of mass of the three dimensional system can be written as

$$\vec{r}_{cm}(X,Y,Z) = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \ldots.}{m_1 + m_2 + \ldots..} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i}$$

Where $\vec{r}_i = x_i \hat{i} + y_i j + z_i k$ is the position vector in three dimensional coordinate system.

1.5. Centre of mass in continuous bodies

When a system contains 'n' number of particles, where the mass and position of each particle is represented by m_i and r_i respectively, then

The centre of mass of the system $\vec{r}_{cm} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$ (1)

Equation (1) represents the summation of the centre of mass of a system. However, this equation will not hold good for continuous bodies, because, a continuous body will have infinitesimal small region. Therefore, instead of summation, we need to integrate the equation(1) for obtaining the centre of mass of continuous bodies.

Let us consider the mass of the one such small region 'dm' and its position 'r'. If the elemental mass m_i is arbitrarily very small in the region i.e., if m_i tends to zero, then equation (1) will become an integral over the entire volume of the body.

$$\vec{r}_{cm} = \lim_{M_i \to 0} \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{\int \vec{r} dm}{M}$$

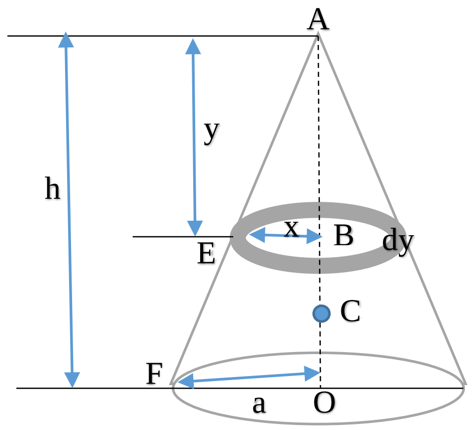
(or) $\vec{r}_{cm} = \frac{\int \vec{r} dm}{M}$ (2)

Equation (2) represents the centre of mass of continuous bodies.

1.5 a centre of mass of a solid cone

Let us consider a solid circular cone of base radius a and height h. let ρ be the density of the material of the cone. If the solid cone is homogeneous, then its mass

$$m = \frac{1}{3}\pi a^2 h.\rho$$



The centre of mass lies on the axis of symmetry AO. The cone is connected to made up of large number of circular discs, each of thickness *dy*.

Let us consider one such elementary disc of radius *x* at a distance *y* from the vertex *A* of the solid cone. The mass of this elementary disc is

$$dm = \rho(\pi x^2)\, dy \qquad (1)$$

From the figure, $\frac{x}{a} = \frac{y}{h}$

(or) $x = \frac{a}{h} y$

$$\therefore dm = \rho.\pi.\left(\frac{a}{h} y\right)^2 .dy \qquad (2)$$

Now from equation (1), we have for the distance of centre of mass on the axis of symmetry *AO* as measured from the vertex *A* as

$$Y_{CM} = \frac{1}{M} \int y.dm \qquad (3)$$

Substituting equation (2), in (3), we get

$$Y_{CM} = \frac{1}{M} \int y.\rho.\pi.\left(\frac{a}{h} y\right)^2 .dy$$

(or) $$Y_{CM} = \frac{\rho.\pi.a^2}{M.h^2} \int y^3.dy \qquad (4)$$

Where the limits of *y* is taken from *y* = 0 to *y* = *h* to cover the entire solid cone filled with such elementary discs. It gives

$$Y_{CM} = \frac{\rho.\pi.a^2}{M.h^2} \left[\frac{y^4}{4}\right]_0^h$$

$$Y_{CM} = \frac{\rho.\pi.a^2}{M.h^2} \times \frac{h^4}{4} = \frac{\rho.\pi.a^2.h^2}{M}$$

But M = Total mass of the solid cone = $\frac{1}{3}\pi a^2 h.\rho$

Hence, $Y_{CM} = \frac{\rho.\pi.a^2.h^2.3}{4.\pi.a^2.h.\rho}$

The CM of cone from its vertex Y_{CM} is written as R_{CM}

$$R_{CM} = \frac{3}{4}h$$

Thus, CM of a solid cone is at a distance of ¾ h from vertex of the cone along the axis.

Centre of mass of a triangular lamina

The medians of the triangle are axes of symmetry in the base of triangular sheets. We simply draw any two medians of the triangle which intersect at a point. This point is the centre of mass of the triangular body (Fig).

We know that the medians bisect each other in the ratio of 2:1 the position of centre of mass on any medians is obtained by dividing that median in the ratio 2:1 the larger portion being towards the vertex. That point is the centre of mass.

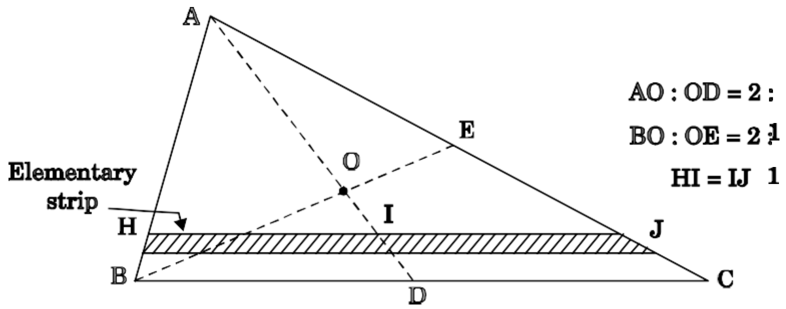


Fig. The point of intersection of the medians of a triangle gives the position of centre of mass

However, the position of centre of mass can also be calculated by assuming the triangle to be made up of large number of strips parallel to one side of the triangle and placed one above the other as shown in the figure.

Centre of mass of some regular objects

Figure shows the centre of masses of some regular shaped homogeneous rigid bodies.

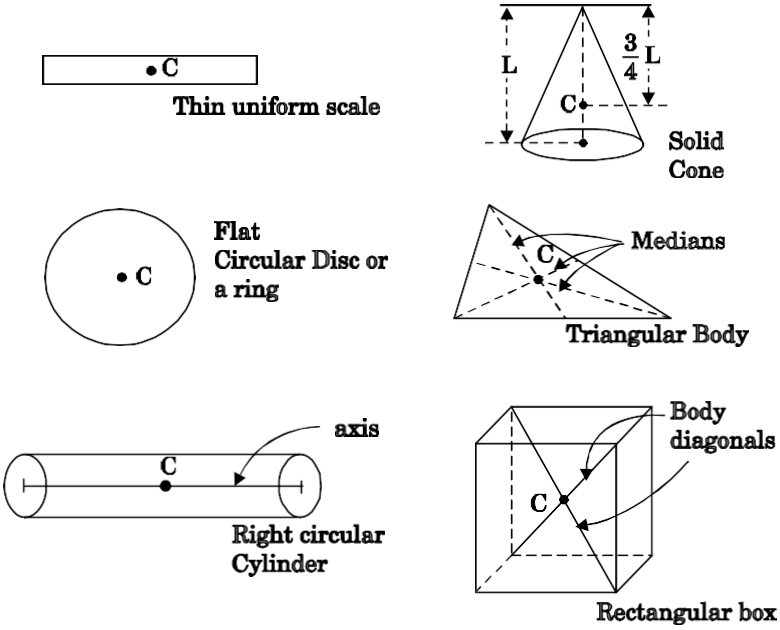
For a rigid body, the centre of mass is a point at a fixed position with respect to the body as a whole. Depending on the shape of the body and the way the mass is distributed in it, the centre of mass is a point may or may not be within the body.

If the shape is symmetrical and the mass distribution is uniform, we can usually find the location of the centre of mass quite easily.

For a long thin rod of uniform cross section and density, the centre of mass is at the geometrical centre.

For a thin circular plane ring, It is again at the geometrical centre of the circle.

For a flat circular disc or rectangle, again the centre of mass is at the geometrical centre.



1.6. Motion of the centre of mass

The motion of the centre of mass is nothing but the force required to accelerate the system of particles with respect to the centre of mass

The motion of centre of mass is nothing but the force required to accelerate the system of particles with respect to the centre of mass.

The motion of the centre of mass shall be obtained as follows:

Let us consider an external force ‘F’ acting on the system of particles along the x-axis.

The centre of mass of the system along x-axis shall be written as

$$x_{cm} = \sum_i \frac{m_i x_i}{m_i}$$

(or) $x_{cm} \sum_i m_i = \sum_i m_i x_i$

Since $\sum m_i = M$, we can write

$$Mx_{cm} = m_1 x_1 + m_2 x_2 + \ldots.. \qquad (1)$$

Differentiating equation (1) with respect to time, we get

$$M\frac{dx_{cm}}{dt} = m_1\frac{dx_1}{dt} + m_2\frac{dx_2}{dt} + ...$$

Differentiating once again with respect to time, we get

$$M\frac{d^2x_{cm}}{dt^2} = m_1\frac{d^2x_1}{dt^2} + m_2\frac{d^2x_2}{dt^2} + ... \qquad (2)$$

Since acceleration is $a = \frac{d^2x}{dt}$, therefore, equation (2), becomes

$$Ma_{cm} = m_1a_1 + m_2a_2 + ... \qquad (3)$$

According to Newton's second law, we know that F = m a

Hence, equation (3) is rewritten as

$$F_{cm} = F_1 + F_2 + ...$$

(or) $F_{cm} = \sum_i F_i$ (4)

Equation (4) represents the force acting on the centre of mass which is equal to the sum of the forces that acting on the system of particles. This force is required to move the particles with respect to the centre of mass (or) so called motion of the centre of mass.

S.No.	Shape of the body	Position of centre of mass
1.	uniform rod	Middle point of rod
2.	Circular disc	Centre of the disc
3.	Circular ring	Centre of the ring
4.	Sphere	Centre of sphere
5.	Hollow sphere	Centre of sphere
6.	Cylinder	Middle point of the axis
7.	Cubical Block	Point of intersection of diagonals joining opposites corners
8.	Plane lamina	Point of intersection of two diagonals
9.	Cone of pyramid	On line joining the apex to the centre of the base of the cone at a distance 1/4th of the length of this line.
10.	Triangular plane lamina	Point of intersection of medians of triangle.

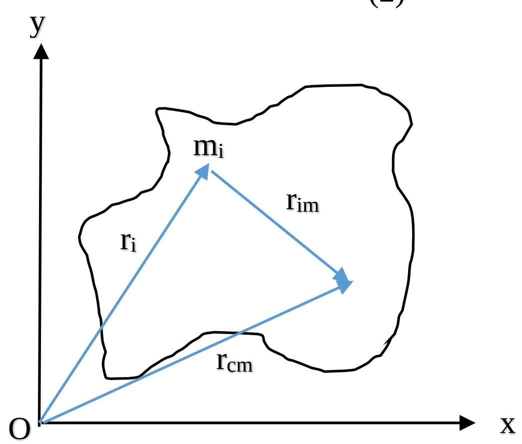
1.7. Kinetic energy of system of particles

Let us consider a multi-particle system with 'n' number of particles in which each particle is moving with some velocity. Let r_i be its displacement and v_i be the velocity of i^{th} particle at any instant of time as shown in figure

Then, the kinetic energy of the i^{th} particle shall be written as $E_K = \sum_i \frac{1}{2} m_i v_i^2$ (1)

If V_{cm} is the velocity of centre of mass with respect to the origin 'O' and v_{im} is the velocity of i^{th} particle with respect to centre of mass. Then the velocity of the i^{th} particle can be written as

$v_i = v_{cm} + v_{im}$ (2)



substituting equation (2) in equation (1), we get

$$E_K = \sum_i \frac{1}{2} m_i (v_{cm} + v_{im})^2$$

(or) $$E_K = \sum_i \frac{1}{2} m_i (v_{cm}{}^2 + v_{im}{}^2 + 2 v_{cm} v_{im})$$

(or) $$E_K = \sum_i \frac{1}{2} m_i v_{cm}{}^2 + \sum_i \frac{1}{2} m_i v_{im}{}^2 + 2 \sum_i \frac{1}{2} v_{cm} v_{im}$$

(or) $$E_K = \sum_i \frac{1}{2} m_i v_{cm}{}^2 + \sum_i \frac{1}{2} m_i v_{im}{}^2 + \sum_i v_{cm} v_{im}$$ (3)

Here, $\sum_i m_i = M$ and The total momentum with respect to centre of mass of the system is,

$$\sum_i m_i v_{im} = 0$$

Therefore, equation (3) becomes

$$E_K = \frac{1}{2} M v_{cm}{}^2 + \frac{1}{2} \sum_i m_i v_{im}^2 + 0$$

(or) $$E_K = \frac{1}{2} M v_{cm}{}^2 + \frac{1}{2} \sum_i m_i v_{im}^2$$ (4)

Equation (4) represents the kinetic energy of the system of the particles.

Here, $\frac{1}{2} M v_{cm}{}^2$ term represents the kinetic energy of the centre of the mass of the system and

$\frac{1}{2}\sum_i m_i v_{im}^2$ represents the sum of kinetic energy of all particles (moving with centre of mass) with respect to the origin.

1.8. Types of motion

In general, there are two types of motion namely rotational and translational motions.

(i) Translational motion

Here, a body moves in a straight line in which all the constituent particles move along parallel straight lines and will undergo equal displacements in equal intervals of time i.e., all the particles in the body will have same velocity an acceleration.

Example: Movement of car, A coin moving over a carom board, An apple falling from a tree, etc.,

(ii) Rotational motion

Here, a body moves about a fixed axis an each particle describes concentric circles about the axis. Though different particles at different points of the body will have different linear velocities, but they will have same angular velocity.

Example: Motion of a door about its hinges, pulleys in vehicles, rotation of blades in a fan, etc.,

1.9. Rotation of rigid bodies

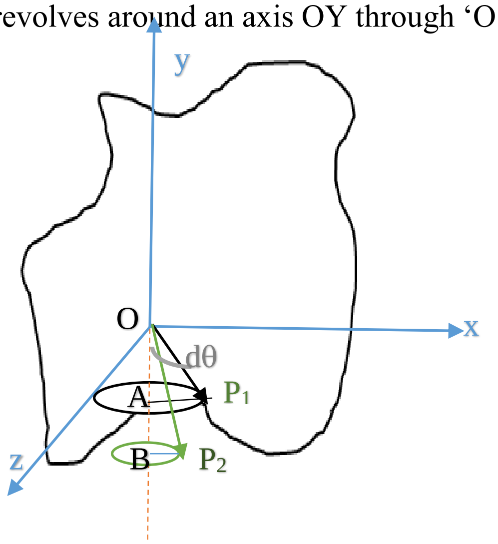
A rigid body is an object which has definite shape and size and does not change due to external force. In otherwords, rigid body can be defined as an extended object in which the distance between particles is not altered during its motion.

Rotational motion

A rotational motion in a rigid body may be considered as a stationary motion and here, the rotation is caused by a couple acting on the body. Its state can be changed only by applying a couple (or) a set of couples.

Explanation

Let us consider a rigid body, which revolves around an axis OY through 'O' as shown in figure.



Let us consider two particles say P_1 and P_2 which revolves in a circular path, about the point A and B respectively. Here, it is found that the centre of each circle lies on OY and the radii of these circles (AP_1 and BP_2) will be equal to perpendicular distance from the axis OY.

We know that in rotational motion, though the particles will have different linear velocities, they will have same angular velocity. Therefore all the particles will rotate through an angle dθ in a small interval of time *dt*

Therefore, Angular velocity $\omega = \frac{d\theta}{dt}$

The corresponding angular acceleration is $\alpha = \frac{d\omega}{dt}$

1.10 Rotational kinematics

It describes the inter relationship between the angular displacement, angular velocity and angular acceleration with respect to the time.

It describes the rotational motion of the particles without considering the mass (or) force affect the rotation. Kinematics of rotational motion for constant angular acceleration with respect to an axis of rotation is analogue to kinematics of linear motion.

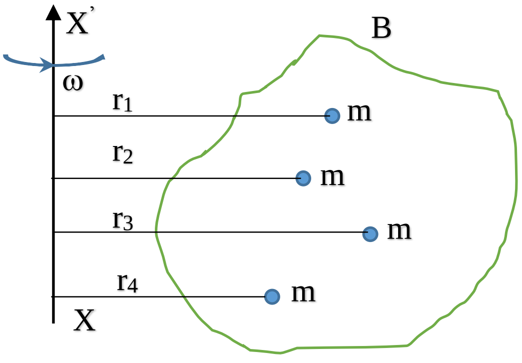
The equation governing the linear motion and rotation motion with various relationship between displacement, velocity, acceleration and time are provided in the table as follows:

Sl. No.	Linear motion	Rotational motion
1	$V_f = u_i + at$	$\omega_f = \omega_i + \alpha t$
2	$S = v_i t + ½ at^2$	$\theta = \omega_i t + ½ \alpha t^2$
3	$V_f^2 = v_i^2 + 2aS$	$\omega_f^2 = \omega_i^2 + 2 \alpha \theta$
	Here v_i - Initial velocity at $t = 0$ v_f - Final velocity at t a - Acceleration t - Time S - Displacement	Here ω_i - Initial angular velocity at t = 0 ω_f - Final angular velocity at t α - Angular acceleration t - Time θ - Angular displacement

1.11 Rotational Kinetic Energy

Let us consider a rigid body rotating about an axis XX' with constant angular velocity 'ω' as shown in figure. All particles in rigid body have the same angular velocity 'ω' but with different linear velocity 'v' varies with radial distance from the axis XX'.

Let $v_1, v_2, \ldots v_i$, be the linear velocities of the particles of mass $m_1, m_2, \ldots\ldots m_i$, rotating about the axis of rotation at distance $r_1, r_2, \ldots\ldots r_i$, respectively.



The kinetic energy of the particle with mass $m_1 = \frac{1}{2} m_1 v_1^2$

The kinetic energy of the particle with mass $m_2 = \frac{1}{2} m_2 v_2^2$

The kinetic energy of the particle with mass m_i = $\frac{1}{2}m_i v_i^2$

Then, the total K.E. of all the particles will corresponds to the K.E. of the body.

$$Total K.E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + ... + \frac{1}{2}m_i v_i^2 \qquad (1)$$

Since all the particles move with same angular velocities (ω) but with different linear velocities (v_1, v_2, ...v_i) at different distances (r_1, r_2, ...r_i) from the axis of rotation

We can write $v_1 = r_1\omega$; $v_2 = r_2\omega$; .... $v_i = r_i\omega$

Equation (1) becomes,

$$Total K.E = \frac{1}{2}m_1 r_1^2\omega^2 + \frac{1}{2}m_2 r_2^2\omega^2 + ... + \frac{1}{2}m_i r_i^2\omega^2$$

(or) Total K.E. = $\frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$ (2)

Since I is the moment of inertia of body about the XX' axis and is given by

$$I = \sum_i m_i r_i^2 \qquad (3)$$

Equation (2), becomes

Total Kinetic energy = $\frac{1}{2}I\omega^2$ (4)

Equation (4) represents the rotational kinetic energy of the particles in a rigid body.

1.12 Moment of Inertia

Moment of inertia of a body about an axis is define as the summation of the product of the mass and square of the perpendicular distance of different particles of the body from the axis of rotation.

Unit: kgm^2

Concept

According to Newton's first law of motion, a body at rest will remain at rest while a body in uniform motion along a straight line will move continuously unless an external force disturbs it. The property due to which a body does not change its state of rest or motion is called inertia.

For the motion in a straight line, inertia depends on the mass of the body. i.e., if the mass is more, then the inertia will be more. However, when a body moves about an axis, the kinetic energy of its rotation not only depend on its mass and angular velocity, but also depends on the axis about which the rotation is taking place.

If we want to rotate a particle or a body for an angle ‘θ’, we need to overcome the system’s ‘angular inertia’ which is often called moment of inertia (it is not just a mass). Thus, the angular inertia not only depends on the mass, but also depends on the square of the distances of particle from the axis of rotation.

Proof

Let us consider a rigid body ‘B’ which consists of ‘n’ number of particles located at different distances from the axis of rotation $XX^{'}$ as shown in figure.

Therefore, the moment of inertia of the first particle $I_1 = m_1r_1^2$

The moment of inertia of the second particle $I_2 = m_2r_2^2$

Therefore, we get the moment of inertia of the entire rigid body by summing the moment of inertia of all particles.

$$\therefore I = \sum_i m_i r_i^2 \qquad (1)$$

Equation (1) represents the moment of inertia of a rigid body.

1.13 Radius of gyration

If the whole mass of the rigid body ‘M’ is assumed to be concentrated at a distance ‘K’ from the axis of rotation, then $I = M K^2$

Here $M = \sum m_i$ and K is the radius of gyration

Definition

The radius of gyration is defined as the distance from the axis of rotation to the point where the entire mass of the body is assumed to be concentrated.

If the rigid body consists of *n* particles of equal mass *m* then the moment of inertia is $I = \sum m r_i^2$

(or) $I = mr_1^2 + mr_2^2 + \ldots\ldots\ldots\ldots\ldots + mr_i^2$

Multiply and divide by *n* on RHS, we have

$$I = nm\left[\frac{r_1^2 + r_2^2 + \ldots. + r_i^2}{n}\right]$$

(or) $I = M K^2$

Where M = *nm* is the mass of the body and

$K = \left[\frac{r_1^2 + r_2^2 + \ldots. + r_i^2}{n}\right]$ is the radius of gyration about a given axis. This K is the rott mean square of the constituent particles in a body from the given axis.

Unit of K is metre

Radius of gyration depends on size, shape, position, configuration of axis of rotation an distribution of mass of body with respect to the axis of rotation.

1.14 Theorems of moment of inertia

The moment of inertia not only depends on the rotation of axis but also depends on the orientation of the body with respect to the axis, which is different for different axis of the same body. Based on the orientation of the boy and with respect to the rotating axis, moment of inertia shall be calculated for various bodies by using the following theorems

(1) Parallel axis theorem (2) Perpendicular axis theorem.

1.15 Parallel axis theorem

It states that moment of inertia with respect to any axis is equal to the sum of moment of inertia with respect to a parallel axis passing through the center of mass and the product of mass and square of the distance between the parallel axis.

Proof

Let us consider a body of mass *M* for which the centre of mass acts as *G*. Let AA' be an axis parallel to XX' passing through G. Let *'x'* be perpendicular distance between the parallel axis AA' and XX' as shown in figure. The body consists of *'n'* number of particles with different masses and at different distances from the XX' axis. Let m_i be the mass of one such particle in the body, located at a distance r_i from the XX' axis.

The moment of inertia of this particle with respect to XX' axis is

$$dI_{xx'} = m_i r_i^2 \tag{1}$$

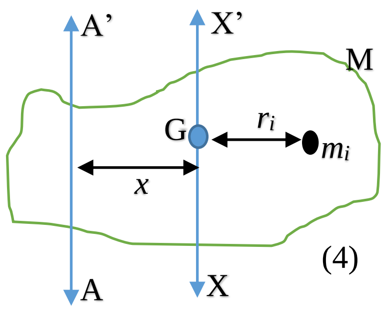
Therefore, the moment of inertia of the entire body with respect to XX' axis is

$$I_{xx} = \sum dI_{XX'} = \sum m_i r_i^2 \tag{2}$$

Similarly, the moment of inertia of this particle with respect to AA' axis is

$$dI_{XX'} = m_i (r_i + x)^2 \tag{3}$$

The moment of inertia of the entire body with respect to AA' axis is



$$I_{AA'} = \sum dI_{AA'} = \sum m_i (r_i + x)^2$$

$$I_{AA'} = \sum dI_{AA'} = \sum m_i ({r^2}_i + x^2 + 2r_i x)$$

$$\therefore I_{AA'} = \sum m_i {r^2}_i + \sum m_i x^2 + 2x \sum m_i r_i \tag{4}$$

According to centre of mass for a rigid boy $\sum m_i r_i = 0$ (r_i has both positive and negative values, so they cancel with eachother.). Further $M = \sum m_i$ (5)

Therefore, from equations (2) an (5) we can write equation (4) as

$$I_{AA'} = I_{XX'} + Mx^2 \qquad (6)$$

Equation (6) represents the parallel axis theorem.

1.16 Perpendicular axis theorem

It states that the moment of inertia of a thin plane body with respect to an axis perpendicular to the thin plane surface is equal to the sum of the moments of inertia of a thin plane with respect to two perpendicular axis lying in the surface of the plane and these three mutually perpendicular axes meet at a common point.

Proof

Let us consider the thin plane body of mass *M* and three mutually perpendicular axes XX', YY' and ZZ' passing through the point 'O'. Let YY' & ZZ' axes lie in the surface of the thin plane and XX' axis lies perpendicular to plane surface as shown in figure.

Let m_i be the mass of one such particle in the body located at a distance r_i from the point 'O'.

The moment of inertiz of the thin plate with respect to XX' axis is

$$dI_{XX'} = m_i r_i^2 \qquad (1)$$

The moment of inertia of the entire body with respect to the axis XX' is

$$I_{XX'} = \sum m_i r_i^2 \qquad (2)$$

From figure, we can write $r_i^2 = y_i^2 + z_i^2$ (3)

Substituting equation (3) in equation (2), we get

$$I_{XX'} = \sum m_i (y_i^2 + z_i^2)$$

$$\therefore I_{XX'} = \sum m_i y_i^2 + \sum m_i z_i^2 \qquad (4)$$

We know that moment of inertia of a thin plane with respect to YY' axis is

$$I_{YY'} = \sum m_i y_i^2$$

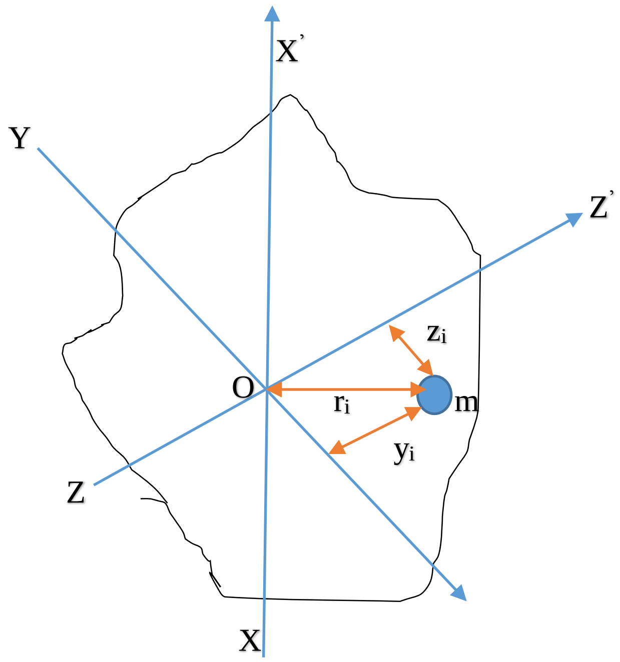
Similarly, the moment of inertia of a thin plate with respect to ZZ' axis is

$$I_{ZZ'} = \sum m_i z_i^2$$

Hence, equation (4) becomes

$$I_{XX'} = I_{YY'} + I_{ZZ'} \qquad (5)$$

Equation (5) represents the perpendicular axis theorem.



1.17 Moment of inertia of continuous bodies

When a body contains 'n' number of particles, where the mass of each particle is represented by m_1, m_2, ……… m_i and its position is represented by r_1, r_2,…..r_i with respect to the rotation axis, then

The moment of inertia of the body $I = \sum_i m_i r_i^2$ (1)

Equation (1) represents the summation of moment of inertia of a system. However, this equation will not hold well for a continuous body, because a continuous body will have infinitesimal small regions.

Therefore, instead of summation, we need to integrate the equation (1) for obtaining the moment of inertia of continuous bodies.

Let us consider the mass of one such small region '*dm*' and its position is '*r*'. If the elemental mass m_i is arbitrarily very small in the region ($m_i \rightarrow 0$), then equation (1) will become an integral over the entire volume of the body.

$$\therefore I = \lim_{m_i \rightarrow 0} \sum_i m_i r_i^2 = \int r^2 dm \qquad (2)$$

Equation (2) represents the moment of inertia of continuous body.

This method is used to find the moment of inertia for various bodies with different shapes. For example, Circular ring, Circular disc, Solid cylinder, Hollow cylinder, Solid sphere, Hollow sphere, etc.,

1.18 moment of inertia of thin uniform rod

Position 1

About an axis through its centre of mass and perpendicular to its length

Let PQ be a thin uniform rod of length l & mass M. The rod is free to rotate abut an axes XX' perpendicular to tis length and passing through the centre of mass 'O'.

Mass per unit length of the rod (linear density) $m = \frac{M}{l}$ (1)

Consider a small element dx at a distance x from 'O'

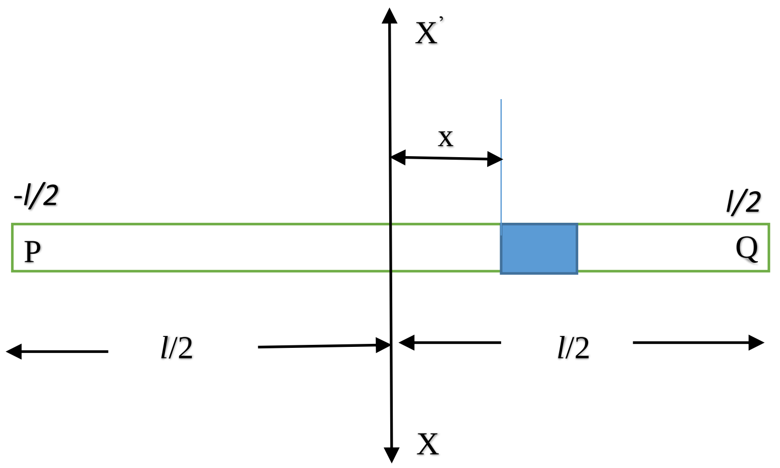
Mass of the element (M) = $m.\ dx$

Moment of inertia of this element about XX' = mass x (distance)^2

$$= m\ dx\ .\ x^2 \qquad (2)$$

The rod consists of number of such elements of length dx . Hence the moment of inertia I of the rod about XX' is obtained by integrating equation (1) between x = $-l/2$ to x = $l/2$.

$$\therefore I = \int_{-\frac{l}{2}}^{\frac{l}{2}} m.x^2 dx \qquad (3)$$



$$I = m\left[\frac{x^3}{3}\right]_{-\frac{l}{2}}^{\frac{l}{2}}$$

$$I = m\left[\frac{\frac{l^3}{8} + \frac{l^3}{8}}{3}\right]$$

(or) $I = m\left[\frac{l^3}{12}\right]$

(or) $I = ml\left[\frac{l^2}{12}\right]$

(or) $I = \left[\frac{Ml^2}{12}\right]$ (4)

Where M = $m\ l$

About an axis passing through one end of the rod and perpendicular to its length

Let PQ be a thin uniform rod of length l an mass M. O is its centre. As the rod is uniform, its centre and centre of gravity coincides. XX' is an axis passing through O and perpendicular to the length of the rod.

Moment of inertia of the rod about XX' = $\left[\frac{Ml^2}{12}\right]$ (5)

Let AA' be an axis passing through one end P an perpendicular to the length of the rod. Let I be the moment of inertia of the rod about this axis AA'

By parallel axis theorem,

$I_{AA'} = I_{xx'} + Mx^2$ (6)

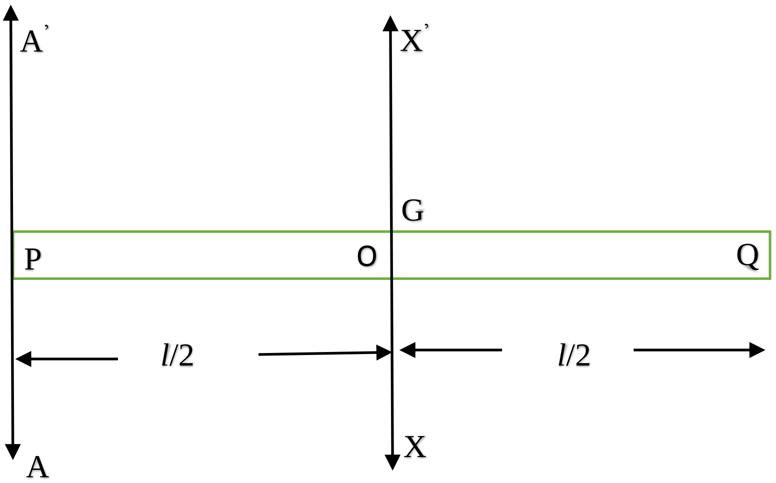
Here the distance x = *l/2*, hence, substituting this and equation (5) in (6), we get

$$I_{AA'} = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2$$

(or) $$I_{AA'} = \frac{Ml^2}{12} + \frac{Ml^2}{4}$$

(or) $$I_{AA'} = \frac{4Ml^2}{12}$$

(or) $$I_{AA'} = \frac{Ml^2}{3}$$



1.19 moment of inertia of a circular ring

Let us find the moment of inertia of a circular ring with rotating axis at various points.

Position1

Rotating axis is passing through the centre of mass (ring centre) and perpendicular to the ring plane

Let us consider a circular ring with radius 'R' and mass 'M' rotating about an axis passing through the centre of ring 'O' as shown in figure. Let us consider a elemental portion of the ring (*dl*) at the circumference of the ring (*L*) and the mass of the elemental ring is *'dm'*

Therefore, the moment of inertia of the elemental ring is given by

$$dI = (dm)R^2 \quad (1)$$

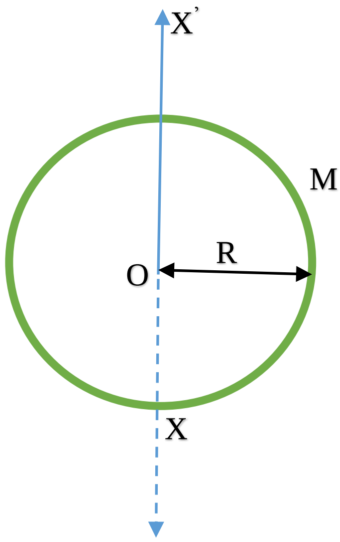
Here, the mass of an elemental portion '*dm*' of the ring is

Mass (*dm*) = Length mass density (μ) X Length of the elemental portion of ring (*dl*)

$$\therefore dm = \mu dl \quad (2)$$

We know, the length mass density of the ring is

$$\mu = \frac{\text{Mass(M)}}{\text{Circumferencial length(L)}} = \frac{M}{2\pi R} \qquad (3)$$



Where R is the radius of the ring

Substituting equation (3) in (2), we get

$$dm = \frac{M}{2\pi R} dl \qquad (4)$$

Substituting equation (4) in (1), we get

$$dI = \frac{M}{2\pi R} dl \times R^2$$

(or) $$dI = \frac{MR}{2\pi} dl \qquad (5)$$

Since the circular ring is a continuous body, we can get the moment of inertia of the circular ring by integrating equation (5) within the limits of 0 to 2πR.

$$\therefore \int dI = \int_0^{2\pi R} \frac{MR}{2\pi} dl$$

(or) $$\int dI = \frac{MR}{2\pi} \int_0^{2\pi R} dl$$

(or) $$I = \frac{MR}{2\pi} [l]_0^{2\pi R}$$

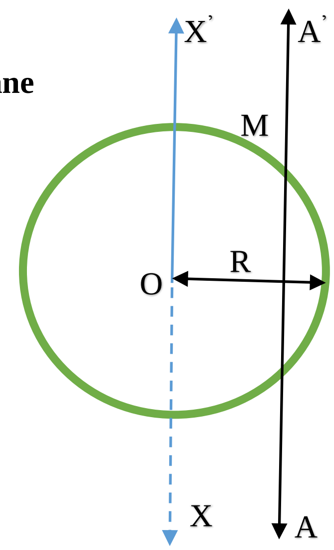
(or) $$I = \frac{MR}{2\pi} [2\pi R]$$

(or) $$I = MR^2$$

Therefore, the moment of inertia of the circular ring when the rotating axis passing through centre of mass is $I = M R^2$. (6)

Position 2

Rotating axis at the edge of the ring and perpendicular to the ring plane



Let AA' be the rotation axis at the edge of the ring which is perpendicular to the ring plane as shown in figure. Here, we can see that XX' axis is passing through the centre of mass of the ring is parallel to AA' axis.

Based on parallel axis theorem, the moment of inertia with respect to AA' axis is given by $I_{AA'} = I_{XX'} + MR^2$ (7)

Using equation (6), $I_{XX'} = MR^2$ (8)

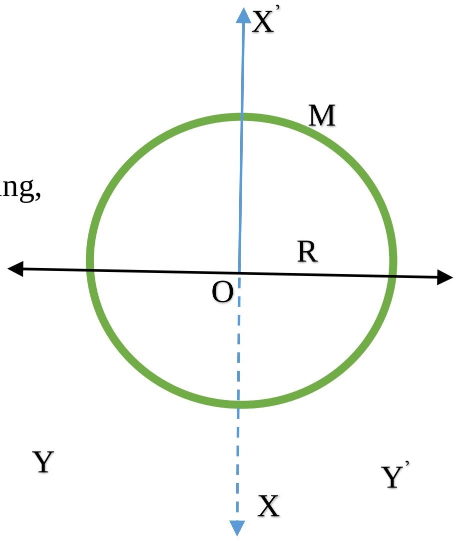
Substituting equation (8) in (7), we get

$I_{AA'} = MR^2 + MR^2$

Therefore, $I_{AA'} = 2MR^2$ (9)

Position 3

Rotating axis is passing through the diameter of the ring



Let YY' be the rotating axis passing through the diameter of the ring, which is perpendicular to XX' as shown in figure

Based on perpendicular axis theorem, we can write

$I_{XX'} = I_{YY'} + I_{ZZ'}$ (10)

Here for circular disc, $I_{ZZ'} = I_{YY'}$

Therefore, equation (10) can be written as

$I_{XX'} = I_{YY'} + I_{YY'}$

(or) $I_{XX'} = 2I_{YY'}$

(or) $I_{YY'} = \frac{I_{XX'}}{2}$ (11)

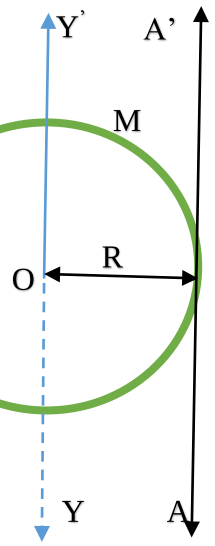
Using equation (6), we can write $I_{XX'} = MR^2$, hence equation (11) becomes,

$$I_{YY'} = \frac{MR^2}{2} \quad (12)$$

Equation (12) represents the moment of inertia when the rotating axis is passing through the diameter of the ring.

Position 4

Rotating axis at the edge of the ring and parallel to ring plane



Let AA' be the rotating axis at the edge of the ring and parallel to the ring plane.

Let YY' be the axis that passes through the diameter of the ring, which is parallel to AA' as shown in figure

Based on parallel axis theorem, the moment of inertia with respect to AA' axis is given by $I_{AA'} = I_{YY'} + MR^2$ (13)

Using equation (12), we can write $I_{YY'} = \frac{MR^2}{2}$ (14)

Substituting equation (14) in (13), we get

$$I_{AA'} = \frac{MR^2}{2} + MR^2$$

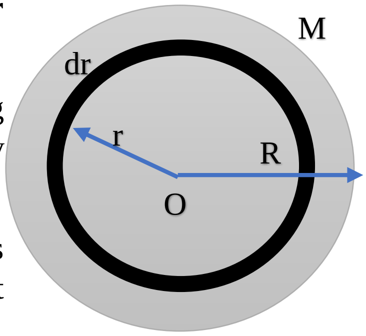
(or) $I_{AA'} = \frac{3}{2}MR^2$ (15)

Equation (15) is the moment of inertia when the rotating axis is at edge of the ring and parallel to ring plane.

1.20 Moment of inertia of a circular disc

Position 1

Rotating axis is passing through the centre of mass and perpendicular to the disc plane



Let us consider a circular disc with radius *R* rotating about an axis passing through the centre of the disc *O*. Let the mass of the disc *M* be uniformly distributed all over the surface area of the disc.

The disc shall be assumed to contain infinitesimally small rings. Let us consider one such ring of the mass *dm* and thickness *dr*, which is located at a distance *r* from the centre of the disc *O* as shown in figure.

Therefore, The moment of inertia of a small ring is given by

$dI = (dm)\, r^2$ (1)

Here, the mass of small ring *(dm)* with radius *r* is given by

Mass (dm) = Surface density *x* circumference of the ring *x* Thickness of the ring

$\therefore dm = (\sigma).(2\pi r).(dr)$ (2)

We know that, the surface mass density for the disc

$\sigma = \frac{Mass(M)}{Area(A)} = \frac{M}{\pi R^2}$ (3)

Substituting equation (3) in equation (2), we get

$$dm = \frac{M}{\pi R^2} 2\pi r.dr$$

$\therefore dm = \frac{2M}{R^2} r.dr$ (4)

Substituting equation (4) in equation (1), we get

$$dI = \frac{2M}{R^2} r.dr.r^2$$

(or) $dI = \frac{2M}{R^2} r^3.dr$ (5)

Since the circular disc is a continuous body, we can get the moment of inertia of the entire disc by integrating equation (5) within the limits 0 to R,

$$\therefore \int dI = \int_0^R \frac{2M}{R^2} r^3 dr$$

$$\text{(or)} \quad I = \frac{2M}{R^2}\int_0^R r^3 dr$$

$$\text{(or)} \quad I = \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_0^R$$

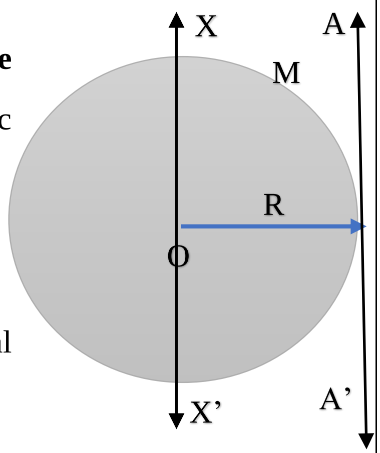
$$\text{(or)} \quad I = \frac{2M}{R^2}\cdot\frac{R^4}{4}$$

The moment of inertia of a circular disc $I = ½ MR^2$ (6)

Equation (6) represents the moment of inertia of a circular disc when the rotation axis is passing through the centre of mass.

Position 2

Rotating axis at the edge of the disc and perpendicular to the disc plane



Let XX' and AA' axis are parallel and both the axis are perpendicular to disc surface as shown in figure.

Based on parallel axis theorem

$I_{AA''} = I_{XX'} + MR^2$ (7)

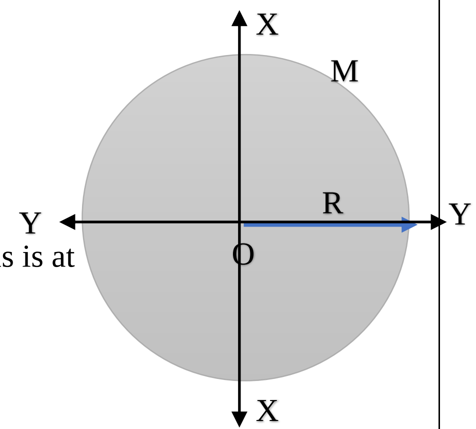
Here, $I_{AA'}$ is the moment of inertia of the circular disc fir which the rotational axis is the edge of the disc.

Using equation (6), we can write $I_{XX'} = ½ MR^2$ (8)

Substituting equation (8) in (7), we get

$I_{AA'} = ½ MR^2 + MR^2$

$I_{AA'} = \frac{3}{2} MR^2$ (9)



Equation (9) represents the moment of inertia, when the rotational axis is at the edge of the disc.

Position 3

Rotating axis is passing through the diameter of the disc

Let YY' be the rotating axis passing through the diameter of the disc, which is perpendicular to XX' as shown in figure

Based on perpendicular axis theorem, we can write

$I_{XX'} = I_{YY'} + I_{ZZ'}$ (10)

Here for circular disc, $I_{ZZ'} = I_{YY'}$

Therefore, equation (10) can be written as

$I_{XX'} = I_{YY'} + I_{YY'}$

(or) $I_{XX'} = 2I_{YY'}$

(or) $I_{YY'} = \frac{I_{XX'}}{2}$ (11)

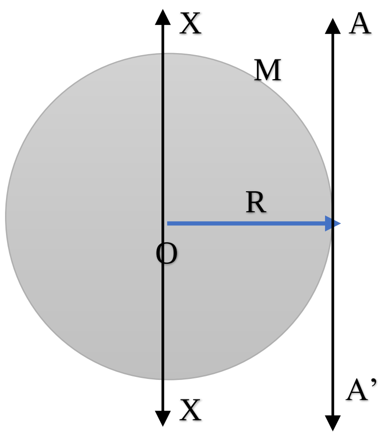
Using equation (6), we can write $I_{XX'} = ½ MR^2$, hence equation (11) becomes,

$$I_{YY'} = \frac{1}{4}MR^2 \quad (12)$$

Equation (12) represents the moment of inertia, when the rotational axis is passing through the diameter of the disc.

Position 4

Rotating axis at the edge of disc and parallel to disc plane



Let YY' and AA' axes are parallel to each other and also parallel to disc surface as shown in figure.

Based on parallel axis theorem, $I_{AA''} = I_{XX'} + MR^2$ (13)

Substituting (13) in (12), we get

$$I_{AA'} = MR^2 + \frac{1}{4}MR^2$$

$$\therefore I_{AA'} = \frac{5}{4}MR^2 \quad (14)$$

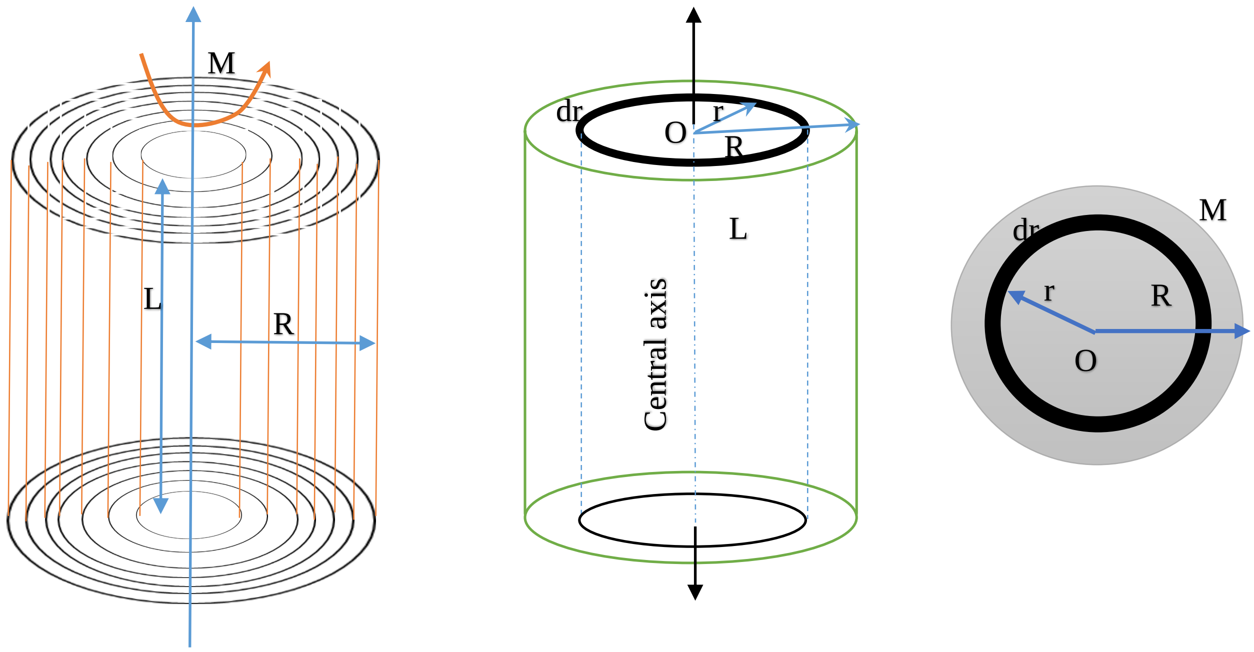
Equation (14) represents the moment of inertia, when the rotational axis is at the edge of the disc an parallel to the plane.

1.21 Moment of inertia of a solid cylinder

Let us consider a solid cylinder of mass M , length L an radius R which contains infinitesimally thin cylinders as shown in figure.

Here the mass is uniformly distributed all over the solid cylinder, rotating about the central axis. Let us consider one such thin cylinder having mass dm thickness dr and length L located at a distance r from central axis of the cylinder as shown in figure

The top view of solid cylinder (of radius R) and thin cylinder (of radius r) are shown in figure



The moment of inertia of the thin cylinder is given by

$dI = (dm)\, r^2$ (1)

Here, the mass of the thin cylinder *(dm)* is

Mass (dm) = Volume density x Area x Length

(or) dm = Volume density x circumference x thickness x length

Therefore, $dm = \rho \,.\, 2\pi r.\, dr \,.\, L$ (2)

We know that, the surface mass density for the disc

$$\sigma = \frac{Mass(M)}{Area(A)} = \frac{M}{\pi R^2 L} \quad (3)$$

Substituting equation (3) in equation (2), we get

$$dm = \frac{M}{\pi R^2 L} 2\pi r.dr.L$$

$$\therefore dm = \frac{2M}{R^2} r.dr \quad (4)$$

Substituting equation (4) in equation (1), we get

$$dI = \frac{2M}{R^2} r.dr.r^2$$

(or) $dI = \frac{2M}{R^2} r^3.dr$ (5)

Since the circular disc is a continuous body, we can get the moment of inertia of the entire disc by integrating equation (5) within the limits 0 to R,

$$\therefore \int dI = \int_0^R \frac{2M}{R^2} r^3 dr$$

$$\text{(or)}\ \ I = \frac{2M}{R^2} \int_0^R r^3 dr$$

$$\text{(or)}\ \ I = \frac{2M}{R^2} \left[\frac{r^4}{4} \right]_0^R$$

$$\text{(or)}\ \ I = \frac{2M}{R^2} . \frac{R^4}{4}$$

The moment of inertia of a circular disc $I = ½ MR^2$ (6)

Equation (6) represents the moment of inertia of a solid cylinder with respect to central axis.

1.22. Moment of inertia of a hollow cylinder

Let us consider a hollow cylinder of inner radius R_1 an outer radius R_2 with length *L* which is rotating about the central axis as shown in figure.

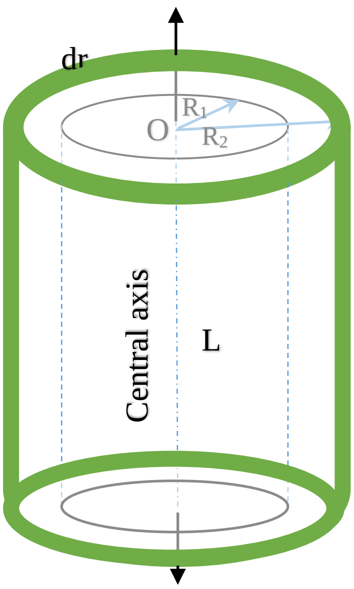
Here the mass is uniformly distributed all over the solid cylinder, rotating about the central axis. Let us consider one such thin cylinder having mass *dm* thickness *dr* and length *L* located at a distance *r* from central axis of the cylinder as shown in figure

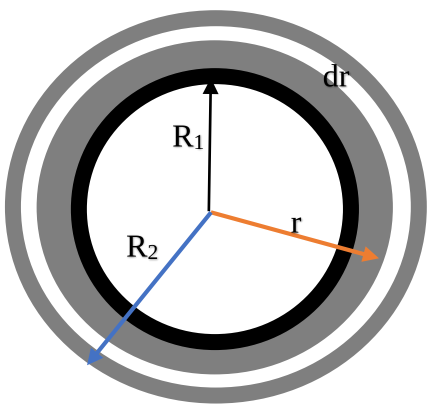
The top view of solid cylinder (of radius *R*) and thin cylinder (of radius *r*) are shown in figure.

Therefore, the moment of inertia of this thin layer of cylinder is given by $dI = (dm)\, r^2$ (1)

Here, the mass of a thin cylinder (*dm*) is

Mass (*dm*) = Volume density x Area x Length





(or) dm = Volume density x circumference x thickness x length

Therefore, $dm = \rho \,.\, 2\pi r .\, dr \,.\, L$ (2)

We know that, the surface mass density for the disc

$$\sigma = \frac{Mass(M)}{Area(A)} = \frac{M}{\pi(R_2{}^2 - R_1^2)L} \quad (3)$$

Substituting equation (3) in equation (2), we get

$$dm = \frac{M}{\pi(R_2^2 - R_1^2)L} 2\pi r.dr.L$$

$$\therefore dm = \frac{2M}{(R_2^2 - R_1^2)} r.dr \quad (4)$$

Substituting equation (4) in equation (1), we get

$$dI = \frac{2M}{(R_2^2 - R_1^2)} r.dr.r^2$$

(or) $$dI = \frac{2M}{(R_2^2 - R_1^2)} r^3.dr \quad (5)$$

Since the circular disc is a continuous body, we can get the moment of inertia of the entire disc by integrating equation (5) within the limits 0 to R,

$$\therefore \int dI = \int_{R_1}^{R_2} \frac{2M}{(R_2^2 - R_1^2)} r^3 dr$$

(or) $$I = \frac{2M}{(R_2^2 - R_1^2)} \int_{R_1}^{R_2} r^3 dr$$

(or) $$I = \frac{2M}{(R_2^2 - R_1^2)} \left[\frac{r^4}{4}\right]_{R_1}^{R_2}$$

(or) $$I = \frac{2M}{(R_2^2 - R_1^2)} . \frac{(R_2^4 - R_1^4)}{4}$$

(or) $$I = \frac{1}{2} . \frac{M(R_2^2 - R_1^2).(R_2^2 + R_1^2)}{(R_2^2 - R_1^2)}$$

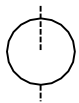
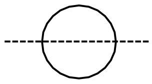
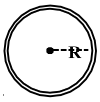
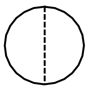
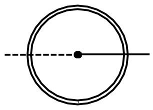
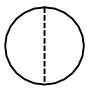
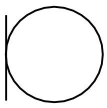
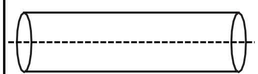
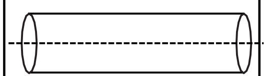
The moment of inertia of a circular disc I = ½ $M(R_2^2 + R_1^2)$ (6)

Equation (6) represents the moment of inertia of a hollow cylinder with respect to central axis.

If the wall of hollow cylinder is very thin, then $R_1 \cong R_2$, which is equal to R,

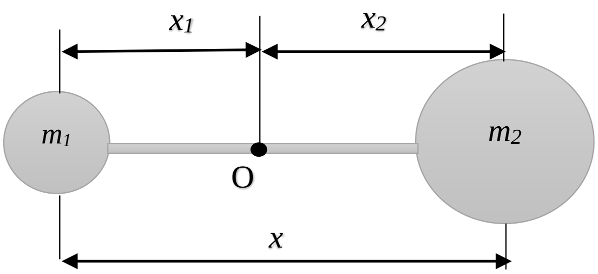
Then Moment of inertia of thin wall hollow cylinder is $I = \frac{1}{2} M (R^2 + R^2)$

$$(or)\ I = MR^2$$

Body	Location Axis	Figure	MI
A thin uniform rod of length l	Passing through the C.G and $\perp r$ to the length		$\frac{M l^2}{12}$
A thin uniform rod of length l	Passing through one end and $\perp r$ to the length		$\frac{M l^2}{3}$
Circular thin ring of radius R	Passing through the centre and $\perp r$ to the plane		MR^2
Circular thin ring of radius R	About any diameter		$\frac{MR^2}{2}$
Circular thin ring of radius R	About a tangent in the plane of ring		$\frac{3}{2} MR^2$
Circular thin circular disc of radius R	About an axis through it centre		$\frac{MR^2}{4}$
Circular thin circular disc of radius R	About any diameter		$\frac{MR^2}{4}$
Solid sphere of radius R	About any diameter		$\frac{2}{5} MR^2$
Solid sphere of radius R	About any tangent		$\frac{7}{5} MR^2$
Solid cylinder of radius R and length l	Passing through CM and $\perp r$ to the length		$M\left[\frac{l^2}{12} + \frac{R^2}{4}\right]$
Solid cylinder of radius R and length l	About symmetry axis		$\frac{MR^2}{2}$

1.23. Moment of inertia of rigid diatomic molecule

Let us consider a rigid diatomic molecule containing two atoms of masses m_1 and m_2 separated by a distance x. Let this diatomic molecule be considered as a system connected by a weightless rigid rod as shown in figure. The centre of mass of the system (diatomic molecule) lies between the two atoms and is denoted by the point O. Let x_1 and x_2 be the distance of two atoms from the point O.



Therefore, from figure, we can write $x = x_1 + x_2$ (1)

Since the system is balanced with respect to the centre of mass , we can write

$$m_1 x_1 = m_2 x_2 \quad (2)$$

From equation (1), we can write $x_2 = x - x_1$ (3)

Substituting equation (3) in equation (2), we get

$$m_1 x_1 = m_2 (x - x_1)$$

(or) $m_1 x_1 = m_2 x - m_2 x_1$

(or) $m_1 x_1 + m_2 x_1 = m_2 x$

(or) $(m_1 + m_2) x_1 = m_2 x$

$$\therefore x_1 = \frac{m_2 x}{m_1 + m_2} \quad (4)$$

From equation (1), we can also write $x_1 = x - x_2$ (5)

Similarly, by substituting equation (5) in equation (2), we get

$$m_1 (x - x_2) = m_2 x_2$$

(or) $m_1 x - m_1 x_2 = m_2 x_2$

(or) $m_1 x = m_1 x_2 + m_2 x_2$

(or) $m_1 x = (m_1 + m_2) x_2$

$$\therefore x_2 = \frac{m_1 x}{m_1 + m_2} \quad (6)$$

Moment of inertia

The moment of inertia (I) of a diatomic molecule with respect to an axis passing through centre of mass of the system shall be written as

$$I = m_1 x_1^2 + m_2 x_2^2 \quad (7)$$

Substituting equation (4) and equation (6) in equation (7), we get

$$I = m_1 \left[\frac{m_2 x}{(m_1 + m_2)} \right]^2 + m_2 \left[\frac{m_1 x}{(m_1 + m_2)} \right]^2$$

$$\text{(or)} \quad I = \frac{x^2}{(m_1 + m_2)^2} \left[m_1.m_2^2 + m_2.m_1^2 \right]$$

$$\text{(or)} \quad I = \frac{x^2 (m_1.m_2)}{(m_1 + m_2)^2} \left[m_1 + m_2 \right]$$

$$\text{(or)} \quad I = \frac{(m_1.m_2)}{(m_1 + m_2)} x^2 \quad (8)$$

Since $\mu = \frac{(m_1.m_2)}{(m_1 + m_2)}$ is called the reduced mass of the system, we can write equation (8) as

$$I = \mu x^2 \quad (9)$$

Equation (9) represents the moment of inertia of a diatomic molecule.

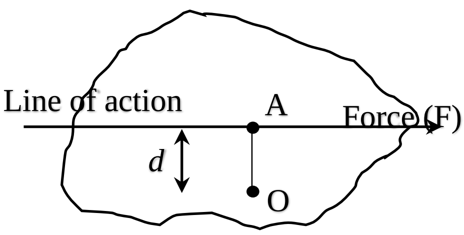
1.24 Moment, Couple and Torque

Moment of force

The moment of force about a point is defined as the product of the magnitude of the force and perpendicular distance from the point to the line of action of force.

Explanation

Let F be the force acting on a body at A as shown in figure.



Then the moment of force F about O is $M_o = F \times d$

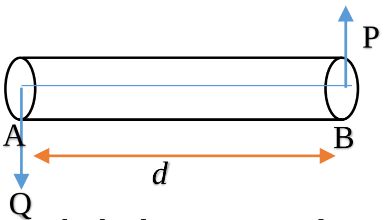
Where d is the perpendicular distance from the point O to the line of action of force F.

Couple

A couple constitutes a pair of two equal and opposite forces acting on a body in such a way that the lines of action of the two forces are not in the same straight line.

Explanation

Let P and Q be the two equal and opposite forces acting on the body AB as shown in figure. Then these two forces from a couple and if the moment of the couple about A is M_A and about B is M_B then we can write



Couple $= M_A = M_B = P \times d$

Torque

Torque is defined as moment of force acting on the body in rotational motion with respect to the fixed point.

Explanation

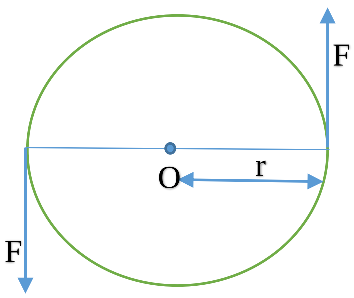
Torque is the rotating force and is equal to the moment of the couple. Torque is the product of one of the forces forming couple and the perpendicular distance from the pivot (or) central point at which the two opposite forces act.

If F is the force acting at a distance r from the centre point O as shown in figure. Then, the product of one of the force forming couple and the perpendicular distance from pivot to force acting point us called torque or moment of force.

Therefore, $torque = force \times radius$

(or) $\tau = F \times r$

Torque is a vector quantity, which is perpendicular to both the direction of force and radius vector.



1.25. Rotational dynamics of rigid bodies

Dynamics of rigid bodies

The dynamics of rigid bodies is the study of effect of external force and couples and its variation with respect to the rigid body.

Concept of rotational dynamics

We know dynamics is the movement of rigid body under the force, which depends on where the force is acting and the state of restriction of the object. If the object has no restriction and force is acting through the centre of gravity, then the movement of the object is purely translational as explained by Newton's law of motion.

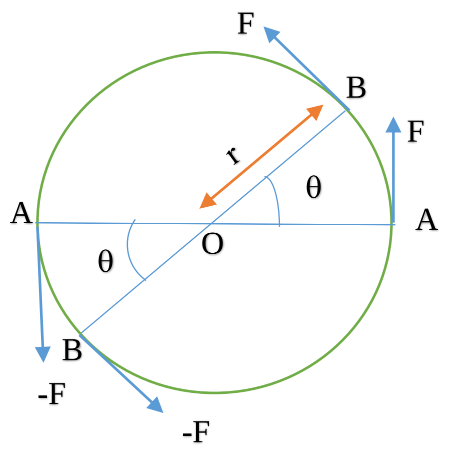
If the object is under the restriction (if the rigidly fixed at one point called pivot) and if the force is acting in such a way that the line of force is not passing through the pivot, then the movement of the object is purely rotational with respect to pivot. This is the concept of rotational dynamics.

Explanation

Let us consider two equal and opposite forces F and $-F$ acting tangentially with respect to the pivot O on the rim of a circular disc from the extremities of diameter as shown in figure. It forms a couple. If the couple rotates through a small angle θ, then, the distance moved by the force F in rotating the body through an angle θ = Length of the arc AB.

Workdone by two forces constitutes a couple = $2\,F\,r\,\theta$

Here the length of the arc $AB = r\,\theta$



Therefore, workdone by a single force = $F\,r\,\theta$

Here $F\,r$ is the moment of the couple (or) torque (τ)

Therefore, workdone by the torque = $\tau\theta$

If L is the angular momentum of the rotating body, then the relation between torque and angular momentum shall be written as Torque $\tau = \frac{dL}{dt}$

Note:

If the object rotation is anti-clock wise, the direction of torque is outward. If the object rotation is clock wise, then the direction of torque is inward.

Rotational dynamics of rigid bodies

The rotational dynamics of rigid bodies are described by the laws of kinematics and the applications of Newton's laws of linear motion and rotational motion

Eg: (1) Torsional pendulum

(2) Double pendulum

(3) Gyroscope

In rotational dynamics the solutions of equations of motion are used to find the position, velocity, momentum, acceleration, etc., of the individual components of the system and course the overall system itself as a function.

1.26. Newton's laws for rotational motion

Newton's first law

An object continue in its state of rest or uniform rotation with a constant angular velocity until it is acted up on by a non zero net torque.

Newton's second law

When an external torque is applied to an object, the torque produces an angular acceleration, which is directly proportional to the torque and inversely proportional to the moment of inertia of the object.

Proof

From Newton's law of linear motion, we can write

F = m . a (1)

We can convert the above linear motion equation (1) to a rotational motion equation by multiplying the radius on both sides.

Therefore, $r\,F = m\,a\,r$ (2)

Here, Torque (τ) = $r\,F$ and acceleration $a = \frac{dv}{dt}$,

Therefore, we can write equation (2) as, $\tau = m\frac{dv}{dt}r$ (3)

Since $v = \omega r$, we can write equation (3) as $\tau = m\frac{d(r\omega)}{dt}r$

$$\text{(or)} \quad \tau = mr^2\frac{d\omega}{dt} \qquad (4)$$

Since I = $m\,r^2$ and Angular acceleration $\alpha = \frac{d\omega}{dt}$

We can write equation (4) as Torque $\tau = I\,\omega$

$$\text{(or)} \quad \text{Angular acceleration } (\alpha) = \frac{\text{Torque } (\tau)}{\text{Moment of inertia } (I)}$$

Hence Newton's II law is proved.

1.27. Conservation of angular momentum

We know that the relation between torque (τ) and angular momentum (L) is,

Torque $\tau = \frac{dL}{dt}$

If no net external torque is acting on the body i.e., if $\tau_{net} = 0$, then the angular momentum (L) of the body will be constant i.e., if $\tau_{net} = 0 \Rightarrow \frac{dL}{dt} = 0$

Therefore L is constant

The above equation is known as the law of conservation of angular momentum. It shows that the angular momentum of the rigid body is constant at any instant of time *t*, if the net torque is zero. In other words we can say that if the net torque is zero, then the angular momentum in a rigid body will be equal. If L_1 and L_2 are the angular momentum, then we can write

$L_1 = L_2$

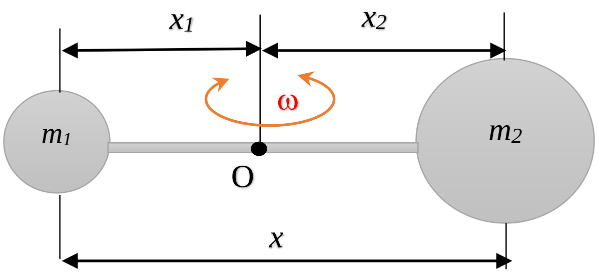
(or) $I_1 \omega_1 = I_2 \omega_2$ = A constant

(or) $I \alpha \frac{1}{\omega}$

Therefore, for a rigid body when the moment of inertia increases, then the angular velocity will decrease (and vice versa). If the external net torque is zero.

1.28. Rotational Energy state of a rigid diatomic molecule

Let us consider a rigid diatomic molecule having two atoms of masses m_1 and m_2 connected by a weightless rod of length *x*. This rigid diatomic molecule rotates with an angular velocity ω with respect to an axis through the centre of mass *O* and is perpendicular to the connecting rod as shown in figure.



We know that the kinetic energy of rotating diatomic molecule is $K.E. = \frac{1}{2} I \omega^2$ (1)

We know that the angular momentum of a rotating body is $L = I\,\omega$ (or) $\omega = \frac{L}{I}$ (2)

Substituting equation (2) in (1), we get $K.E. = \frac{1}{2} I \frac{L^2}{I^2}$

$$\text{(or)}\ K.E. = \frac{L^2}{2I} \qquad (3)$$

We know that the moment of inertia of a rotating diatomic molecule is $I = \mu x^2$ (4)

Substituting equation (4) in (3), we get, Kinetic energy $K.E. = \frac{L^2}{2\mu x^2}$ (5)

Equation (5) represents the classical equation for kinetic energy of a rigid diatomic molecule, in which the energy levels are continuous for all possible values of 'L'.

But according to quantum mechanics, we know that the energy values are discrete.

Based on quantum theory, the angular momentum L shall be written as $L = \sqrt{J(J+1)}.\hbar$ (6)

Where J is the total angular momentum quantum number and its values are 0, 1, 2, 3,… so on.

Substituting equation (6) in equation (5), we get $E_J = \frac{J(J+1).\hbar^2}{2\mu x^2}$ (7)

This equation (7) represents the rotational kinetic energy of a rigid diatomic molecule, quantum mechanically.

Special cases

When J = 0, equation (7) becomes, $E_0 = 0$

When J = 1, equation (7) becomes, $E_1 = \frac{2.\hbar^2}{2\mu x^2} \; (or) E_1 = \frac{\hbar^2}{\mu x^2}$ (8)

When J = 2, equation (7) becomes, $E_2 = \frac{2(3).\hbar^2}{2\mu x^2} \; (or) E_1 = \frac{3\hbar^2}{\mu x^2}$ (9)

From eqn. (8) and (9), $E_2 = 3 E_1$

When J = 3, equation (7) becomes, $E_2 = \frac{3(4).\hbar^2}{2\mu x^2} \; (or) E_1 = \frac{6\hbar^2}{\mu x^2}$ (10)

From eqn. (8) and (10), $E_3 = 6 E_1$

Therefore, In general, $E_J = \frac{J(J+1)}{2} E_1$

From these results, we can confirm that rotational kinetic energy of rigid diatomic molecule is quantized and discrete.

1.29 Gyroscope

The main principle used in gyroscope is the product of angular momentum which is experienced by the torque on the wheel (or) disc is used to produce a gyroscopic precession in the spinning wheel.

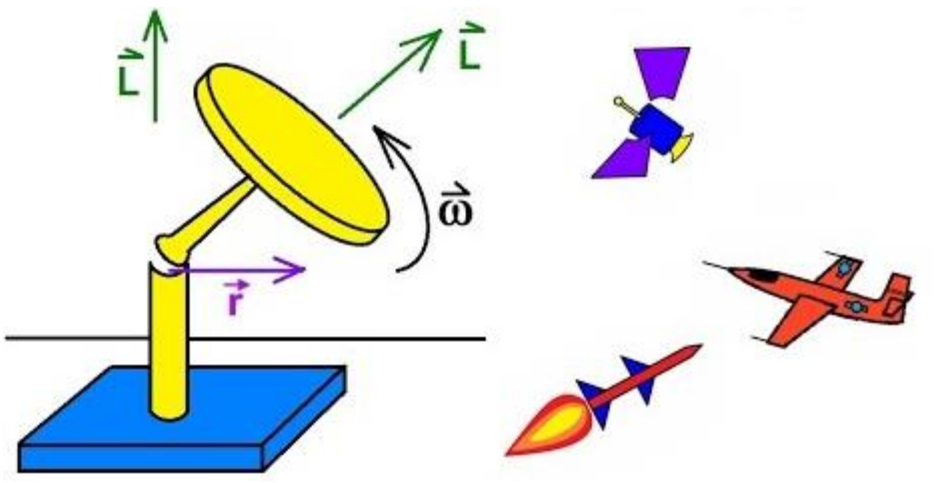
Types

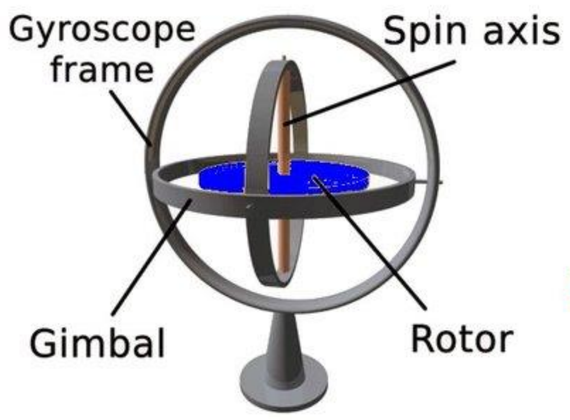
There are different types of gyroscopes

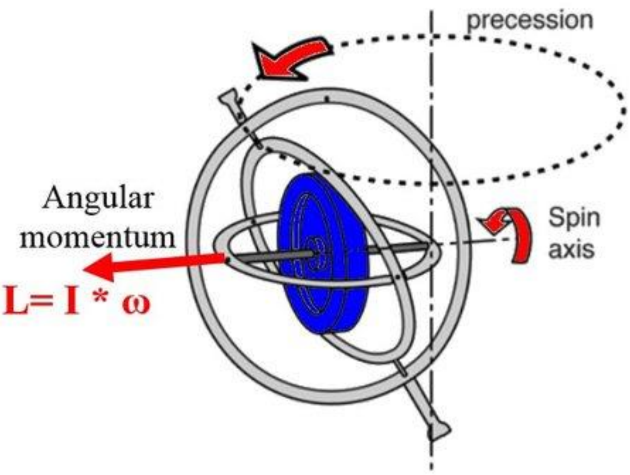
(1) Mechanical gyroscope

(2) Optical gyroscope

(3) Gas bearing gyroscope







Let us discuss about mechanical gyroscope

Design

The gyroscope consists of four main parts as shown in figure.

(i) Rotor (ii) Gimbal (iii) Spinning wheel (iv) Gyroscope frame with base

In gyroscope the massive rotor is fixed on the supporting rings known as gimbals. The rotor will have three degrees of rotation, which will be helpful to alter the following parameters

(i) Angular velocity (ω) (ii) Angular momentum (L) (iii) Torque (τ) of the rotation motion.

The above three parameters are inter related. Here the direction of angular momentum act in the same direction as that of the rotational axis in symmetrical bodies.

Working

Without spinning

If there is no spinning of wheel (i.e., L = 0), the free end only move to horizontal plane (XY plane) due to gravitational force.

With spinning

If there is spinning of wheel, the free end moves towards download direction combined with the spin of the wheel about the axis. Hence, a download force W = *m g* will act at a distance *r* and will produce an angular momentum simultaneously and rotates the spinning wheel along the horizontal plane as shown in figure.

Therefore, gyroscope movement steadily increases depends upon time interval in horizontal direction based on the equation given by

$$\sum \tau = \frac{dL}{dt} \quad \text{(or)} \quad \sum \tau .dt = dL \qquad (1)$$

From equation (10, we can see that the gyroscope experience a net torque and therefore angular momentum must change. Due to constant direction, torque and angular momentum will alter its direction without change of magnitude. As a result, the axis of rotation of wheel does not fall. Thus the gyroscope maintain its orientation even though the base is moved to any place.

Applications

Gyroscopes are used in the following areas:

1. They are used as compass in boats, aeroplanes, air crafts, etc.,
2. It is used in space craft in order to navigate the space craft to the desired target
3. It is used to stabilize the ships, satellites, ballistic missiles, etc.,
4. Gyroscopes along with accelerometers are used in smart phones for providing excellent motion sensing.

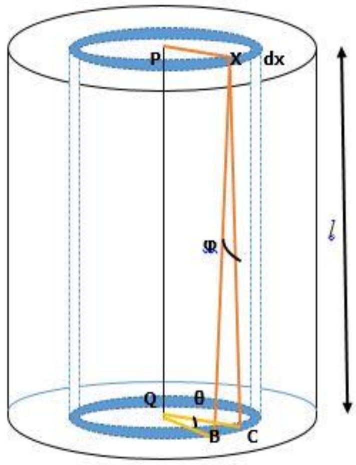
1.30. Twisting couple on a wire

Consider a cylindrical wire of length ***l*** and radius **r** fixed at one end. It is twisted through an angle θ by applying couple to its lower end. Now, the wire is said to be under torsion. Due to elastic property of the wire, an internal restoring couple is setup inside the wire. It is equal and opposite to the external twisting couple. The cylinder is imagined to consist of a large number of thin hollow cylinders.

Consider one such cylinder of radius **x** and thickness **dx. AB** is a line parallel to **PQ** on the surface of this cylinder. As the cylinder is twisted, the line **AB** is shifted to **AC** through an angle **BAC = ϕ**

Shearing Strain = ϕ

Angle of twist at the free end = θ



From the figure, **BC** = $x\,\theta = l\,\theta$ (or) $\phi = \frac{x\theta}{l}$

Rigidity modulus (n) = $\frac{\textbf{Shearing Stress}}{\textbf{Shearing Strain}}$

∴Shearing stress = n x Shearing strain = $n\phi = \frac{nx\theta}{l}$

But, Shearing stress = $\frac{\textbf{Shearing Force}}{\textbf{Area over which the force acts}}$

Shearing Force = Shearing stress x area over which the force acts

Area over which the force acts is $\pi(x+dx)^2 - \pi x^2 = 2\,\pi\,x\,dx$ (neglecting dx^2)

Hence, shearing force $F = \frac{nx\theta}{l} 2\pi x dx$

Twisting couple on a wire

Shearing force $F = \frac{2\pi n\theta}{l} x^2 dx$

∴Moment of this force about the axis **PQ** of the cylinder = Force x perpendicular distance

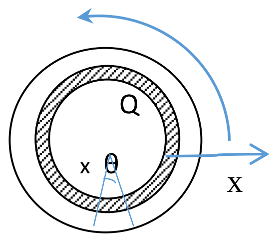
$$= \frac{2\pi n\theta}{l} x^2 dx \times x$$

$$= \frac{2\pi n\theta}{l} x^3 dx$$

The moment of the force acting on the entire cylinder of radius ***r*** is obtained by integrating the above expression between the limits $x = r$ and $x = 0$

Hence, twisting couple $C = \int_0^r \frac{2\pi n\theta}{l} x^3 dx$

$$\frac{2\pi n\theta}{l}\int_0^r x^3 dx = \frac{2\pi n\theta}{l}\left[\frac{x^4}{4}\right]_0^r$$



∴ $C = \frac{\pi n r^4 \theta}{2l}$

In the above equation, if θ = 1 radian, then, we get

Twisting couple per unit twist $\boxed{C = \frac{\pi n r^4}{2l}}$

This twisting couple required to produce a twist of unit radian in the cylinder is called the torsional rigidity for material of the cylinder

1.31. Torsional Pendulum

A torsional pendulum is a pendulum performing torsional oscillations. It is used to find the rigidity modulus of the material of the wire and moment of inertia of a given disc

Description

A torsional pendulum consists of a metal wire suspended vertically with the upper end fixed. The lower end of the wire is connected to the center of a heavy circular disc as shown in figure. When the disc is rotated by applying a twist, the wire is twisted is twisted through an angle θ.

Then, the restoring couple setup in the wire = **Cθ** where **C** is the couple per unit twist.

If the disc is released, it oscillates with angular velocity $\frac{d\theta}{dt}$ in the horizontal plane about the axis of the wire. These oscillations are known as **torsional oscillations.** If $\frac{d^2\theta}{dt^2}$ is the angular acceleration produced in the disc and **I** its moment of inertia about the axis of the wire then,

Applied couple $= I\frac{d^2\theta}{dt^2}$

At equilibrium position, Applied couple = Restoring couple

i.e., $I\frac{d^2\theta}{dt^2} = C\theta$

(or) $\frac{d^2\theta}{dt^2} = -\frac{C}{I}\theta$

This equation represents simple harmonic motion which shows that angular acceleration is proportional to angular displacement θ and is always directed towards the mean position. Hence, the motion of the disc being *simple harmonic motion,*

The time period of the oscillation is given by $T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$

$$= 2\pi\sqrt{\frac{\theta}{\frac{C}{I}\theta}}$$

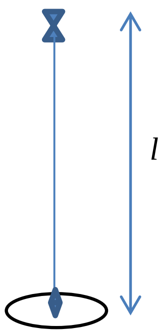
(or)

$$\boxed{T = 2\pi\sqrt{\frac{I}{C}}}$$

Determination of Rigidity modulus of the wire

A circular disc is suspended by a thin wire, whose rigidity modulus is to be determined. The top end of the wire is fixed tightly in a vertical support. The disc is then rotated about its center through a small angle and set it free. It executes torsional oscillations. The time taken for 20 complete oscillations is noted. The experiment is repeated and the mean time period (T) of oscillation is found out.

The length *l* of the wire is measured. This length is then changed and the experiment is repeated for five or six different lengths of wire are measured and tabulated. The disc is removed and its mass and diameter are measured



The time period of oscillation is $\mathbf{T = 2\pi \sqrt{\frac{I}{C}}}$

(or) $T^2 = 4\pi^2 \frac{I}{C}$

Substituting couple per twist $\mathbf{C} = \frac{\pi n r^4}{2l}$

$T^2 = 4\pi^2 \frac{I}{\frac{\pi n r^4}{2l}}$

(or) $\boxed{\eta = \frac{8\pi I}{r^4}\left[\frac{l}{T^2}\right]}$ N m^{-2}

Where **I** is moment of inertia of circular disc which is equal to $\frac{MR^2}{2}$

M- Mass of the circular disc; R – Radius of the disc

1.32. Double pendulum

Double pendulum consists of two pendulums in which one pendulum is attached to the end of the other pendulum. If the motion is small then the pendulum behaves as a simple pendulum. If the motion is large then it behaves as a chaotic system.

Description

Let us consider a double pendulum suspended to a point *O* which consists of pendulum-1 of mass m_1 and pendulum-2 of mass m_2 as shown in figure. Let l_1 be the length of pendulum-1 and l_2 be the length of pendulum-2. If the double pendulum is made to oscillate, then both pendulum will oscillate at an angle θ_1 and θ_2 respectively as shown in figure 1.32(a).

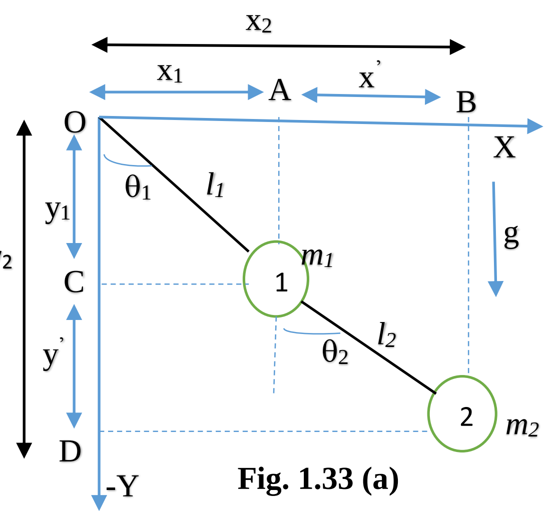


Fig. 1.33 (a)

Derivations

Let us derive the expressions for the displacement, velocity, kinetic energy, potential energy and the Lagrangian of the double pendulum.

Displacement

Let x_1 (OA) and x_2 (OB) be the displacement of pendulum-1 and pendulum-2 respectively, along *x*- axis and let y_1 (OC) and y_2 (OD) be the displacement of pendulum-1 and pendulum-2 respectively, along the negative *y*-axis. Then, from the following figure, we can write

$$\sin\theta_1 = \frac{x_1}{l_1}$$

(or) $x_1 = l_1 \sin\theta_1$ (1)

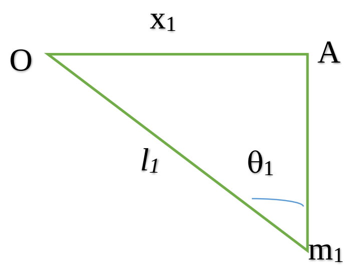


Fig. (b)

Similarly, from second figure, we can write

$$\cos\theta_1 = \frac{-y_1}{l_1}$$

(or) $y_1 = -l_1 \cos\theta_1$ (2)

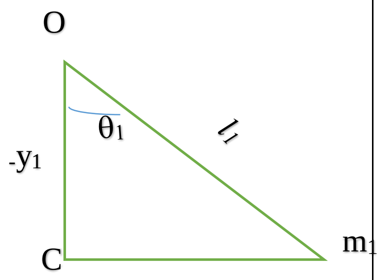


Fig. (c)

Here, the negative sign indicates the –ve y – direction. Since the displacement of pendulum-2 depends on pendulum-1, from figure 1.32, we can write the displacement of pendulum-2 along x-axis as

$x_2 = x_1 + x'$ (3)

From fig. (d), we can write

$$\sin\theta_2 = \frac{x'}{l_2}$$

(or) $x' = l_2 \sin\theta_2$ (4)

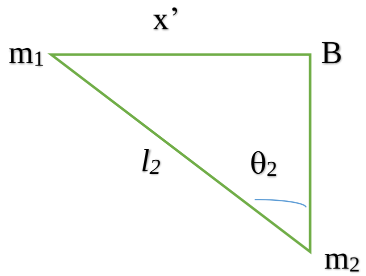


Fig. (d)

Substituting eqn.(4) in (3), we get

$x_2 = x_1 + l_2 \sin\theta_2$ (5)

Substituting equation (1) in equation (5), we get

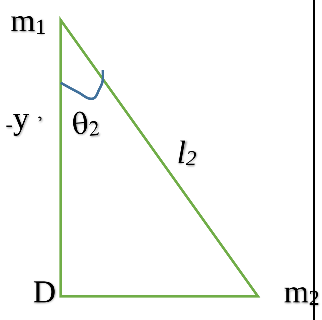
$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$ (6)

Similarly, from fig 1.33(a), we can write the displacement of pendulum-2 along y-axis as

$y_2 = y_1 + y'$ (7)

From Fig. (e), we can write $\cos\theta_2 = \frac{-y'}{l_2}$

(or) $y' = -l_2 \cos\theta_2$ (8)



Substituting equation (8) in equation (7), we get

$$y_2 = y_1 - l_2 \cos\theta_2 \qquad (9)$$

Substituting equation (2) in equation (9), we get

$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2 \qquad (10)$$

Equations (1), (2), (6) and (10) represents the displacement at various positions of the double pendulum.

Velocity

Differentiating equation (1) we get, $V_{x1} = \frac{dx_1}{dt} = \frac{d(l_1 \sin\theta_1)}{dt}$

(or) $V_{x1} = l_1 \cos\theta_1 \frac{d\theta_1}{dt}$

Using the relation $\frac{d\theta_1}{dt} = \dot{\theta}_1$, we can write the above equation as

$$V_{x1} = l_1 \cos\theta_1 \dot{\theta}_1 \qquad (11)$$

Differentiating equation (2), we get

$$V_{y1} = \frac{dy}{dt} = \frac{d(-l_1 \cos\theta_1)}{dt}$$

$$V_{y1} = l_1 \sin\theta_1 \frac{d\theta_1}{dt}$$

(or) $V_{y1} = l_1 \sin\theta_1 \dot{\theta}_1$ (12)

Differentiating equation (6), we get

$$V_{x2} = \frac{dx_2}{dt} = \frac{d(l_1 \sin\theta_1 + l_2 \sin\theta_2)}{dt}$$

(or) $V_{x2} = l_1 \cos\theta_1 \frac{d\theta_1}{dt} + l_2 \cos\theta_2 \frac{d\theta_2}{dt}$

(or) $V_{x2} = l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2$ (13)

Differentiating equation (10) we get

$$V_{y2} = \frac{dy_2}{dt} = \frac{d(-l_1 \cos\theta_1 - l_2 \cos\theta_2)}{dt}$$

(or) $V_{y2} = l_1 \sin\theta_1 \frac{d\theta_1}{dt} + l_2 \sin\theta_2 \frac{d\theta_2}{dt}$

(or) $V_{y2} = l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2$ (14)

Equation (11), (12), (13) and (14) represents the velocity at various positions of the double pendulum.

Kinetic energy

We know, the kinetic energy of the system is

$$T = \sum_{i=1}^{2} \frac{1}{2} m_i (V_{x_i}^2 + V_{yi}^2)$$

(or) $$T = \frac{1}{2} m_i (V_{x1}^2 + V_{y1}^2) + \frac{1}{2} m_i (V_{x2}^2 + V_{y2}^2) \qquad (15)$$

Substituting equation (11), (12), (13) and (14) in (15), we get

$$T = \frac{1}{2} m_1 (l_1^2 \cos^2 \theta_1 \dot{\theta}_1^2 + l_1^2 \sin^2 \theta_1 \dot{\theta}_1^2) + \frac{1}{2} m_2 \left[(l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2)^2 + (l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2)^2 \right] \qquad (16)$$

Equation (16) represents the kinetic energy of the double pendulum.

Potential energy

We know that, the potential energy of the system is

$$V = m_1 g y_1 + m_2 g y_2 \qquad (17)$$

Substituting (2) and (10) in (17), we get

$$V = -m_1 g (l_1 \cos\theta_1 - m_2 g l_1 \cos\theta_1 + l_2 \cos\theta_2) \qquad (18)$$

Equation (18) represents the potential energy of the double pendulum.

Lagrangian

The Lagrangian of the double pendulum is $L = T - V$ (19)

Substituting equation (16) and (18) in (19), we get

$$\left.\begin{aligned} L = &\frac{1}{2} m_1 (l_1^2 \cos^2 \theta_1 \dot{\theta}_1^2 + l_1^2 \sin^2 \theta_1 \dot{\theta}_1^2) + \\ &\frac{1}{2} m_2 \left[(l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2)^2 + (l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2)^2 \right] + \\ &m_1 g l_1 \cos\theta_1 + m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2) \end{aligned}\right\} \qquad (20)$$

Equation (20) represents the Lagrangian equation for double pendulum.

1.33 Introduction to non-linear oscillations

Linear oscillations

A linear Oscillator will oscillate with single frequency in ‘to’ and ‘fro’ (or) up and down motion. Its motion will be sinusoidal and periodic in nature.

Examples (1) Oscillation of pendulum system in a watch (2) Damped oscillator

Non-linear Oscillations

A non-linear oscillator will oscillate with different frequencies in the same time internal (or) in terms of least integer fractions.

Examples (1) Torsional pendulum (or) double pendulum (2) Damped oscillator

Characteristics

(1) In non-linear oscillations, the period of oscillations depends on the amplitude of the oscillations

(2) For some type of non-linearity, the frequency of the oscillator will change with amplitude.

(3) Therefore, the non-linear oscillations will have multiple steady state solutions.

(4) Non-linear oscillations will have jumping phenomena

(5) The non-linear oscillations will have complex (or) irregular motion

(6) These oscillations will possess internal resonances. i.e., different parts of the system will oscillate at different frequencies.

Explanation

Let us consider a non-linear oscillator which oscillates with different frequencies (f_n).

i.e., $f_n = \frac{1}{T_n}$ (1)

Where T_n is the time period.

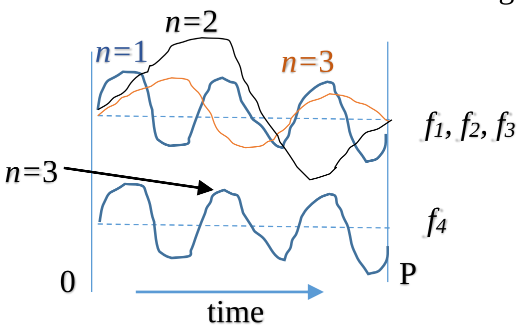
In non-linear oscillator, the time period will be same (or) it be the least integer fractions such as T, T/2, T/3, etc.,

Therefore, we can write, $T_n = \frac{T}{n}$ (2)

Where n is the integer i.e., n = 1, 2, 3,..

Substituting equation (2) in equation (1), we get $f_n = \frac{n}{T}$ (3)

From equation (3), we can conclude that the frequency of the non-linear oscillatory motion is inversely proportional to the time period. For various values of *n* the frequency of oscillations will change and the first four harmonics are shown in figure.



Interference

(1) From figure, we can see that at the mid point, all the harmonics are zero.

(2) The even harmonics will happen for the integer number of cycles and hence it goes positive

(3) The odd harmonics will happen for ½, 3/2, 5/2 etc., number of cycles and hence it goes negative.

(4). Thus the motion of the non-linear oscillator will have complex motion.

QUESTIONS FOR UNIT - 1

Part- A (2 marks)

1. **Define multiparticle dynamics.**
   The study of dynamics of a system which consists of two or more particle is known as multiparticle dynamics.

2. **What is centre of mass?**
   If the mass of the entire particles in the object is concentrated at a particular point, that point is called as centre of mass.

3. **Give three examples for motion of centre of mass**
   (1) Motion of planets and its satellite
   (2) Projectile Trajectory
   (3) Decay of a Nucleus

4. **How centre of mass is determined for rigid body and regular shape?**
   Regular objects
   (1) for a thin long rod of uniform cross section and density. Circular plane ring and rectangle, the centre of mass is at geometrical centre
   Rigid body
   The centre of mass is a point at a fixed position with respect to the object as a whole. Depending on the shape and mass distribution, the centre of mass may or may not be lie within the object.

5. What is the difference between centre of gravity and centre of mass?

The centre of gravity of a body is a point, where the whole weight of the body supposed to be concentrated

The centre of mass of a body is a point, where the whole mass of the body is supposed to be concentrated.

For uniformly cross sectional bodies, the centre of gravity coincides with the centre of mass. However, they do not coincide in objects whose density is not uniform throughout.

6. Define rigid body.

A body which does not unergo any change in shape or volume when external force are applied on it.

7. Write the kinematics of rotational motion

$$\omega = \omega_0 + \alpha t$$

$$\theta = \omega t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

8. Define moment of inertia of a body

The property of a body by which it resists change uniform rotational motion is called rotational inertia or moment of inertia.

9. Define moment of inertia of a rigid body

The moment of inertia of a rigid body about a given axis is the sum of products of masses of its particles and the square of their respective distances from the axis of rotation. i.e., $I = \sum_i m_i r_i^2$

10. What are the physical significance of moment of inertia

- The property which opposes the change in rotational motion of the body is called the moment of inertia.
- Greater is moment of inertia of the body about the axis of rotation, greater is torque required to rotate the body.
- The role of moment of inertia in rotational motion is similar to the role of mass in linear motion.

11. What is radius of gyration?

- The radius of gyration is defined as the distance from the axis of rotation to the point where the entire mass of the body is assumed to be concentrated.
- K is called the radius of gyration of the body about the axis of rotation.
- It is equal to the root mean square distance of all particles from the axis of rotation of the body.

12. State parallel axis theorem

It states that moment of inertia with respect to any axis is equal to sum of moment of inertia with respect to a parallel axis passing through the centre of mass and the product of mass and square of the perpendicular distance between the parallel axis.

13. State perpendicular axis theorem

The moment of inertia of a thin plane boy with respect to an axis perpendicular to the thin plane surface is equal to the sum of the moments of inertia of a thin plane with respect to two perpendicular axes lying in the surface of the plane and these three mutually perpendicular axes meet at a common point.

14. Define angular momentum

Angular momentum of a particle is defined as its moment of linear momentum it is given by the product of linear momentum and perpendicular distance of its line of action from the axis of rotation. It is denoted by $\vec{L}$

15. Define torque

It is defined as moment of force acting on the body in rotational motion with respect to the fixed point.

$\vec{\tau} = \vec{F} \times \vec{r}$

16. State the law of conservation of angular momentum

If net external torque does not act on the body, the angular momentum of the body will be constant. (angular momentum remains conserved)

$\tau_{ext} = 0 \Rightarrow \frac{dL}{dt} = 0$, therefore, L is a constant.

This is known as law of conservation of angular momentum.

17. Prove that the rotational kinetic energy is conserved in the torque free motion of a rigid body.

The rotational kinetic energy is = $\frac{1}{2} I\omega^2$

For torque free motion, the angular velocity is constant. The moment of inertia is time independent parameter, therefore the rotational kinetic energy is conserved if torque is not present.

18. What is gyroscope?

It is a device used for measuring or maintaining orientation and angular velocity. It is a spinning wheel or disc in which the axis of rotation (spin axis) is free to assume any orientation by itself.

19. What are the uses of gyroscopes?

It is used as stabilizers in ships, boats and aeroplanes.

It is used as a compass and gyro-compass which is superior than magnetic compass

It is used in spacecraft in order to navigate the spacecraft to the desired target.

20. What is the torsional pendulum?

A circular metallic disc suspended using a thin wire that executes torsional oscillation is called torsional pendulum.

21. What is double pendulum?

A double pendulum is a pendulum with another pendulum attached to its end.

The pendulum behaves like a linear system for small angles.

When the angles are small in the double pendulum, the system behaves like the linear double spring.

In this case, the motion is determined by simple sine and cosine functions.

On the other hand for large angles, the pendulum is non-linnear and the phase graph becomes much more complex.

22. What is the difference between linear & non-linear oscillations

A linear oscillator will oscillate with single frequency in *to* and *fro* (or) up and down motion. Its motion will be sinusoidal and periodic in nature.

A non linear oscillator will oscillate with different frequencies in the same interval (or) in terms of least integer fractions.

Part –B (16 Marks)

1. Define centre of mass of a system of particles. Derive an expression for centre of mass in a one dimensional system and also discuss about centre mass in three dimensional system.

2. (i) Discuss the centre of mass of continuous bodies
   (ii) Explain the motion of the centre of mass.

3. Derive an expression for kinetic energy of system of particles

4. (i) Derive an expression for the rotational kinetic energy of a rigid body rotating about a fixed axis with an angular velocity ω

5. State and prove the parallel axes theorem for the moment of inertia of a rigid body.

6. State and prove the perpendicular axes theorem for the moment of inertia of plane lamina.

7. Derive an expression for the moment of inertia of a uniform rod.
   (a) About an axis through the centre and perpendicular to its length
   (b) About an axis passing through the end of the rod and perpendicular to its length.

8. Derive an expression for the moment of inertia of a thin ring.
   (a) About an axis through the centre and perpendicular to its plane
   (b) About a diameter.
   (c) About a tangent in the plane of the ring

9. Derive an expression for the moment of inertia of a thin circular disc.
   (a) About an axis through the centre and perpendicular to its plane
   (b) About a diameter.

10. Derive an expression for the moment of inertia of a solid cylinder.
    (a) About an axis through the centre and perpendicular to its length
    (b) About the axis of cylinder.

11. Discuss the moment of inertia of a diatomic molecule.

12. Discuss the rotational energy states of a rigid diatomic molecule.

13. Describe principle, construction and working of gyroscope. Mention its application in various fields.

14. Derive an expression for time period of torsion pendulum. Explain how it is used to rigidity modulus of a wire.

15. Derive an expression for Lagrangian equation for double pendulum with necessary diagram