PH25101 PHYSICS FOR MECHANICAL ENGINEERING

UNIT – I FUNDAMENTAL MECHANICS

Introduction – statics and dynamics – the centre of mass of a system of particles – kinetic energy of a system of particles – Theorems of the moment of inertia – moment of inertia of diatomic molecule – rotational energy levels.

1. Introduction

Mechanics is a branch of physics which deals with the motion of bodies under the action of forces. In elementary mechanics, most of the bodies are assumed to be rigid. But in actual practice, nobody is perfectly rigid. When a stationary body is acted upon by some external forces, then the body may start to rotate (or) move about any point. If the body doesn't move (or) rotate then it is said to be in equilibrium.

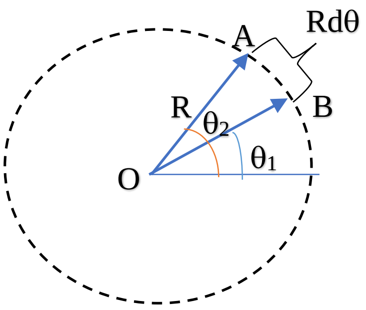
We know that the rigid body is the combination of many particles i.e., multiparticle. Let us discuss the basic definitions relate to mechanics

(1) Angular displacement

Definition

The change in position of the particle moving in a circular path with respect to an angle ($d\theta$) is called angular displacement.

Proof



Let us consider a particle of mass *m* moving in a circular path of radius 'R' with respect to the center of the circle O. At *t=0 sec*, the particle is located at the point *A* and after time interval *t* , it reaches the point *B* as shown in figure.

W.K.T., the angular displacement of a particle is the change in angular Position between two points A and B, which can be measured by the angle (θ_2-θ_1) between the radius vector of these two positions *A* and *B*.

∴ the angle between *A* and *B* is $d\theta = (\theta_2 - \theta_1)$.

Angular displacement $d\theta = (\theta_2 - \theta_1)$. (unit: Radian)

We can write arc length as $AB = l$,

Then the relation between angular displacement ($d\theta$) and linear displacement (l) is given by its arc length as $l = R\, d\theta$.

(2) Angular velocity

The rate of change of angular displacement is called angular velocity

i.e., Angular velocity (ω) = $d\theta$ /dt.   (unit : Rad s^{-1})

The relation between angular velocity (ω) and linear velocity (v) is given by $v = r\ \omega$.

(3) Angular acceleration

The rate of change of angular velocity is called angular acceleration.

i.e., Angular acceleration (α) = $d\omega/dt$ (or) $d^2\theta / dt^2$. (Unit: Rad s^{-2}).

(4) Angular momentum

The moment of inertia times of angular velocity of the particle is called angular momentum.

i.e., Angular momentum $L = I\ \omega$ (Unit : kgm^2s^{-1})

(5) Inertia

It is the tendency of an object to maintain its state of rest or of uniform motion along the same direction. Inertia is a resisting capacity of an object to alter its state of rest and motion (direction and /or magnitude).

1.2. Statics and Dynamics

Statics

It is a branch of science, which deals with the study of a body at rest. Statics is the study of a body at rest. Statics is a study of the equilibrium of bodies under the action of forces, hence, statics is mainly concern with the conditions of equilibrium of stationary bodies

For example:

(i) The equilibrium of the book lying on the table.and the study of different forces acting on it.
(ii) Support reactions of stationary beam subjected to some external loads
(iii) Member forces in a truss, subjected to some external loads, etc.,

Dynamics

It is branch of science which deals with the study of a body in motion. In dynamics, we are mainly concerned with the study of motion of bodies and the effect of forces actin on them.

For example:

(i) Force applied or brakes, when the moving vehicle is brought to rest.
(ii) The path traversed by a particle and the distance covered when it is thrown in space
(iii) The force hit by a cricket bat on ball etc.,

Dynamics is further divided into two subdivisions: Kinematics and Kinetics

Kinematics:

It is a study of a body in motion, without considering the forces, that cause the motion Kinematics is sued to relate displacement, velocity, acceleration, time taken, etc., of the bodies without any reference to the cause of motion.

Kinetics:

It is study of a body in motion, with considering the forces, that cause the motion. Kinetics is used to predict the motion of body caused by given force, or to determine the force required to produce a given motion.

1.2.1 Multiparticle dynamics (Dynamics in a system of particles)

We know dynamics is the study of motion of bodies under the action of forces. Multiparticle dynamics (dynamics in a system of particles) is the study of motion in respect of a group of particles in which the separation between the particles will be very small i.e., the distance between the particles will be negligible.

Explanation

In dynamics, we study the physical parameters by considering an object as a point mass and its shape and size is ignored. But, in real world problems, object will execute rotational and translational motion. For example, if we kick the football, it has both translational and rotational motions. As both the motion depends on the size and shape of the object, both cannot be ignored, even it is negligible. Thus, the study of rotational and translator motion with respect to the system of particles is called multi-particle dynamics.

1.3. Centre of mass

We know that mass is the measure of the body's resistance to change the motion (or) it is measure of inertia of the body. It is a scalar quantity and it is constant.

(i) A system consists of many particles with different masses and different position from the reference point.

(ii) The mass of the system is equal to the sum of the mass of each particle in the system.

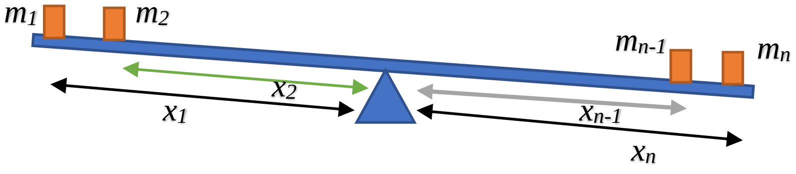
Hence, if the mass of the entire particles of the system is concentrated at a particular point, that point is called centre of mass of the system.

1.3.1. Centre of mass in a one-dimensional system

The system consists of many particles with different positions and different masses. If the mass of the entire particle in the system is concentrated at a particular point, then that point is called centre of mass of the system.

Explanation

Let us consider a fulcrum placed along the x axis which is not at equilibrium position as shown in figure.



Let the position of masses m_1, m_2, m_3, ….., m_{n-1}, m_n be at a distance of x_1,x_2,……..,x_{n-1},x_n respectively from the fulcrum. *The tendency of a mass to rotate with respect to origin or supporting point is called moment of mass.*

The moment of mass for an elemental mass m_n with respect to the fulcrum can be written as $m_n x_n$. If the moments on both sides are equal, then the system is sai to be in equilibrium. Therefore, total moments with respect to the fulcrum shall be written as

$$m_1 x_1 + m_2 x_2 + ... + m_n x_n = \sum_{i=1}^{N} m_i x_i = 0 \qquad (1)$$

If the total moment is equal to zero, then the centre of mass will lie at the supporting point (or) fulcrum and the system is said to be in equilibrium. If the fulcrum is placed at the unbalanced position, then it is shifted to a balanced position (say of distance X) to reach the equilibrium position.

Under equilibrium condition,

$$\sum_{i=1}^{n} m_i x_i - \sum_{i=1}^{n} m_i X = 0$$

(or) $$\sum_{i=1}^{n} m_i x_i = \sum_{i=1}^{n} m_i X$$

(or) $$\sum_{i=1}^{n} m_i x_i = X \sum_{i=1}^{n} m_i$$

(or) $$X = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \qquad (2)$$

Where $\sum_{i=1}^{n} m_i x_i$ is the moment of system and

$\sum_{i=1}^{n} m_i$ is the mass of the system

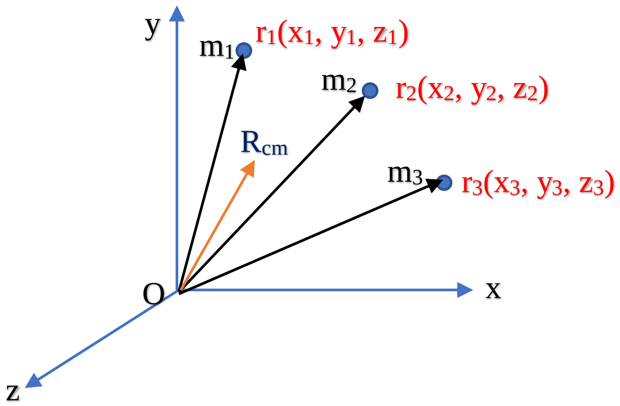
Thus, the system should be move to a distance of X metres in order to attain the balanced position of the system.

The distanced moved to obtain equilibrium position (or) so called the centre of mass in a one dimensional system is given by

$$X = \frac{m_1 x_1 + m_2 x_2 + ...}{m_1 + m_2 + ...} \qquad (3)$$

1.3.2. Centre of mass in three dimensional system

To find the centre of mass in a three dimensional system, let us consider a three dimensional system in which let m_1, m_2, m_3,…. be the masses placed at position vectors $r_1(x_1, y_1, z_1)$, $r_2(x_2, y_2, z_2)$,…. Respectively from the origin 'O' as shown in figure



Here,

(i) The centre of mass along the x – axis,

$$X = \frac{m_1 x_1 + m_2 x_2 + ...}{m_1 + m_2 + ...} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

(ii) The centre of mass along y-axis,

$$Y = \frac{m_1 y_1 + m_2 y_2 + ...}{m_1 + m_2 + ...} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}$$

(iii) The centre of mass along z-axis,

$$Z = \frac{m_1 z_1 + m_2 z_2 + ...}{m_1 + m_2 + ...} = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i}$$

In general, centre of mass of the three-dimensional system can be written as

$$\vec{r}_{cm}(X,Y,Z) = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + ...}{m_1 + m_2 + ...} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i}$$

Where $\vec{r}_i = x_i \hat{i} + y_i j + z_i k$ is the position vector in three-dimensional coordinate system.

1.3.3 Motion of the centre of mass

The motion of the centre of mass is nothing but the force required to accelerate the system of particles with respect to the centre of mass

The motion of centre of mass is nothing but the force required to accelerate the system of particles with respect to the centre of mass.

The motion of the centre of mass shall be obtained as follows:

Let us consider an external force ‘F’ acting on the system of particles along the x-axis.

The centre of mass of the system along x-axis shall be written as

$$x_{cm} = \sum_i \frac{m_i x_i}{m_i}$$

(or) $x_{cm} \sum_i m_i = \sum_i m_i x_i$

Since $\sum m_i = M$, we can write

$$Mx_{cm} = m_1 x_1 + m_2 x_2 + ... \qquad \textbf{(1)}$$

Differentiating equation (1) with respect to time, we get

$$M \frac{dx_{cm}}{dt} = m_1 \frac{dx_1}{dt} + m_2 \frac{dx_2}{dt} + ...$$

Differentiating once again with respect to time, we get

$$M \frac{d^2 x_{cm}}{dt^2} = m_1 \frac{d^2 x_1}{dt^2} + m_2 \frac{d^2 x_2}{dt^2} + ... \qquad (2)$$

Since acceleration is $a = \frac{d^2 x}{dt^2}$, therefore, equation (2), becomes

$$Ma_{cm} = m_1 a_1 + m_2 a_2 + ... \qquad (3)$$

According to Newton’s second law, we know that F = m a

Hence, equation (3) is rewritten as

$$F_{cm} = F_1 + F_2 + ...$$

(or) $F_{cm} = \sum_i F_i$ (4)

Equation (4) represents the force acting on the centre of mass which is equal to the sum of the forces that acting on the system of particles. This force is required to move the particles with respect to the centre of mass (or) so called motion of the centre of mass.

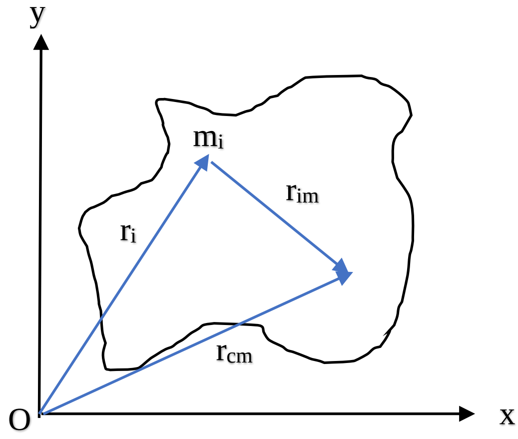
1.4. Kinetic energy of system of particles

Let us consider a multi-particle system with ‘n’ number of particles in which each particle is moving with some velocity. Let r_i be its displacement and v_i be the velocity of i^{th} particle at any instant of time as shown in figure

Then, the kinetic energy of the i^{th} particle shall be written as $E_K = \sum_i \frac{1}{2} m_i v_i^2$ (1)

If V_{cm} is the velocity of centre of mass with respect to the origin ‘O’ and v_{im} is the velocity of i^{th} particle with respect to centre of mass. Then the velocity of the i^{th} particle can be written as

$v_i = v_{cm}+v_{im}$ (2)



substituting equation (2) in equation (1), we get

$$E_K = \sum_i \frac{1}{2} m_i (v_{cm} + v_{im})^2$$

(or) $E_K = \sum_i \frac{1}{2} m_i ({v_{cm}}^2 + {v_{im}}^2 + 2 v_{cm} v_{im})$

(or) $E_K = \sum_i \frac{1}{2} m_i {v_{cm}}^2 + \sum_i \frac{1}{2} m_i {v_{im}}^2 + 2 \sum_i \frac{1}{2} v_{cm} v_{im}$

(or) $E_K = \sum_i \frac{1}{2} m_i {v_{cm}}^2 + \sum_i \frac{1}{2} m_i {v_{im}}^2 + \sum_i v_{cm} v_{im}$ (3)

Here, $\sum_i m_i = M$ and the total momentum with respect to centre of mass of the system is,

$$\sum_i m_i v_{im} = 0$$

Therefore, equation (3) becomes

$$E_K = \frac{1}{2} M {v_{cm}}^2 + \frac{1}{2} \sum_i m_i v_{im}^2 + 0$$

(or) $E_K = \frac{1}{2} M {v_{cm}}^2 + \frac{1}{2} \sum_i m_i v_{im}^2$ (4)

Equation (4) represents the kinetic energy of the system of the particles.

Here, $\frac{1}{2} M {v_{cm}}^2$ term represents the kinetic energy of the centre of the mass of the system and

$\frac{1}{2} \sum_i m_i v_{im}^2$ represents the sum of kinetic energy of all particles (moving with centre of mass) with respect to the origin.

1.5 Moment of Inertia

Moment of inertia of a body about an axis is define as the summation of the product of the mass and square of the perpendicular distance of different particles of the body from the axis of rotation.

Unit: kgm^2

Concept

According to Newton's first law of motion, a body at rest will remain at rest while a body in uniform motion along a straight line will move continuously unless an external force disturbs it. The property due to which a body does not change its state of rest or motion is called inertia.

For the motion in a straight line, inertia depends on the mass of the body. i.e., if the mass is more, then the inertia will be more. However, when a body moves about an axis, the kinetic energy of its rotation not only depend on its mass and angular velocity, but also depends on the axis about which the rotation is taking place.

If we want to rotate a particle or a body for an angle 'θ', we need to overcome the system's 'angular inertia' which is often called moment of inertia (it is not just a mass). Thus, the angular inertia not only depends on the mass, but also depends on the square of the distances of particle from the axis of rotation.

Proof

Let us consider a rigid body 'B' which consists of 'n' number of particles located at different distances from the axis of rotation XX' as shown in figure.

Therefore, the moment of inertia of the first particle $I_1 = m_1r_1^2$

The moment of inertia of the second particle $I_2 = m_2r_2^2$

Therefore, we get the moment of inertia of the entire rigid body by summing the moment of inertia of all particles.

$$\therefore I = \sum_i m_i r_i^2 \qquad (1)$$

Equation (1) represents the moment of inertia of a rigid body.

1.5.1 Radius of gyration

If the whole mass of the rigid body 'M' is assumed to be concentrated at a distance 'K' from the axis of rotation, then $I = M K^2$

Here $M = \sum m_i$ and K is the radius of gyration

Definition

The radius of gyration is defined as the distance from the axis of rotation to the point where the entire mass of the body is assumed to be concentrated.

If the rigid body consists of *n* particles of equal mass *m* then the moment of inertia is

$$I = \sum m r_i^2$$

(or) $I = mr_1^2 + mr_2^2 + \ldots\ldots\ldots\ldots\ldots + mr_i^2$

Multiply and divide by *n* on RHS, we have

$$I = nm\left[\frac{r_1^2 + r_2^2 + \ldots. + r_i^2}{n}\right]$$

(or) $I = M K^2$

Where M = *nm* is the mass of the body and

$$K = \left[\frac{r_1^2 + r_2^2 + \ldots. + r_i^2}{n}\right]$$

is the radius of gyration about a given axis. This K is the rott mean square of the constituent particles in a body from the given axis.

Unit of K is metre

Radius of gyration depends on size, shape, position, configuration of axis of rotation an distribution of mass of body with respect to the axis of rotation.

1.5.3 Theorems of moment of inertia

The moment of inertia not only depends on the rotation of axis but also depends on the orientation of the body with respect to the axis, which is different for different axis of the same body. Based on the orientation of the boy and with respect to the rotating axis, moment of inertia shall be calculated for various bodies by using the following theorems

(1) Parallel axis theorem (2) Perpendicular axis theorem.

1.5.4 Parallel axis theorem

It states that moment of inertia with respect to any axis is equal to the sum of moment of inertia with respect to a parallel axis passing through the center of mass and the product of mass and square of the distance between the parallel axis.

Proof

Let us consider a body of mass *M* for which the centre of mass acts as *G*. Let AA' be an axis parallel to XX' passing through G. Let *'x'* be perpendicular distance between the parallel axis AA' and XX' as shown in figure. The body consists of *'n'* number of particles with different masses and at different distances from the XX' axis. Let m_i be the mass of one such particle in the body, located at a distance r_i from the XX' axis.

The moment of inertia of this particle with respect to XX' axis is

$$dI_{xx'} = m_i r_i^2 \qquad (1)$$

Therefore, the moment of inertia of the entire body with respect to XX' axis is

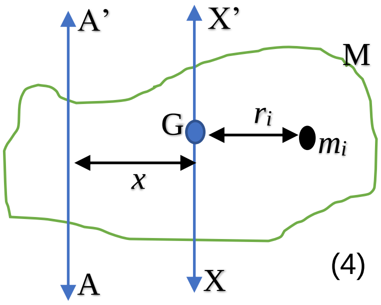
$$I_{xx} = \sum dI_{XX'} = \sum m_i r_i^2 \qquad (2)$$

Similarly, the moment of inertia of this particle with respect to AA' axis is

$$dI_{XX'} = m_i (r_i + x)^2 \qquad (3)$$

The moment of inertia of the entire body with respect to AA' axis is

$$I_{AA'} = \sum dI_{AA'} = \sum m_i (r_i + x)^2$$



$$I_{AA'} = \sum dI_{AA'} = \sum m_i (r^2{}_i + x^2 + 2r_i x)$$

$$\therefore I_{AA'} = \sum m_i r^2{}_i + \sum m_i x^2 + 2x \sum m_i r_i \quad (4)$$

According to centre of mass for a rigid boy $\sum m_i r_i = 0$ (r_i has both positive and negative values, so they cancel with eachother.). Further $M = \sum m_i$ (5)

Therefore, from equations (2) an (5) we can write equation (4) as

$$I_{AA'} = I_{XX'} + Mx^2 \quad (6)$$

Equation (6) represents the parallel axis theorem.

1.5.5 Perpendicular axis theorem

It states that the moment of inertia of a thin plane body with respect to an axis perpendicular to the thin plane surface is equal to the sum of the moments of inertia of a thin plane with respect to two perpendicular axis lying in the surface of the plane and these three mutually perpendicular axes meet at a common point.

Proof

Let us consider the thin plane body of mass *M* and three mutually perpendicular axes XX', YY' and ZZ' passing through the point 'O'. Let YY' & ZZ' axes lie in the surface of the thin plane and XX' axis lies perpendicular to plane surface as shown in figure.

Let m_i be the mass of one such particle in the body located at a distance r_i from the point 'O'.

The moment of inertiz of the thin plate with respect to XX' axis is

$$dI_{XX'} = m_i r_i^2 \quad (1)$$

The moment of inertia of the entire body with respect to the axis XX' is

$$I_{XX'} = \sum m_i r_i^2 \quad (2)$$

From figure, we can write $r_i^2 = y_i^2 + z_i^2$ (3)

Substituting equation (3) in equation (2), we get

$$I_{XX'} = \sum m_i (y_i^2 + z_i^2)$$

$$\therefore I_{XX'} = \sum m_i y_i^2 + \sum m_i z_i^2 \quad (4)$$

We know that moment of inertia of a thin plane with respect to YY' axis is

$$I_{YY'} = \sum m_i y_i^2$$

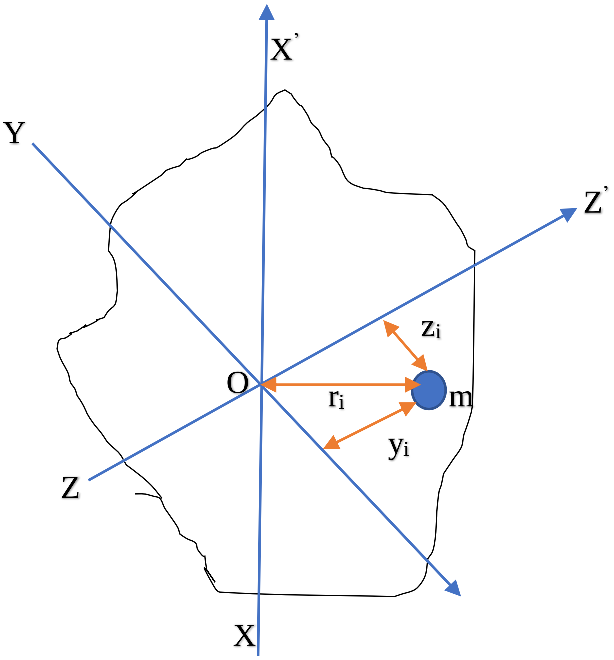
Similarly, the moment of inertia of a thin plate with respect to ZZ' axis is

$$I_{ZZ'} = \sum m_i z_i^2$$

Hence, equation (4) becomes

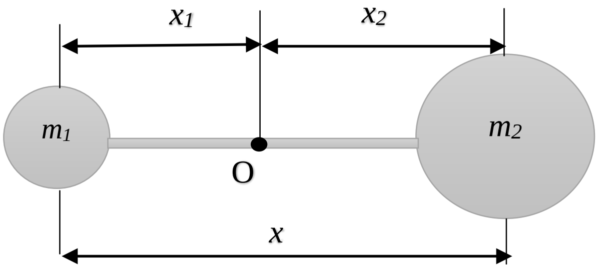
$$I_{XX'} = I_{YY'} + I_{ZZ'} \qquad (5)$$

Equation (5) represents the perpendicular axis theorem.



1.5.6. Moment of inertia of rigid diatomic molecule

Let us consider a rigid diatomic molecule containing two atoms of masses m_1 and m_2 separated by a distance x. Let this diatomic molecule be considered as a system connected by a weightless rigid rod as shown in figure. The centre of mass of the system (diatomic molecule) lies between the two atoms and is denoted by the point O. Let x_1 and x_2 be the distance of two atoms from the point O.



Therefore, from the figure, we can write $x = x_1 + x_2$ (1)

Since the system is balanced with respect to the centre of mass , we can write

$$m_1 x_1 = m_2 x_2 \quad (2)$$

From equation (1), we can write $x_2 = x - x_1$ (3)

Substituting equation (3) in equation (2), we get

$$m_1 x_1 = m_2 (x - x_1)$$

(or) $m_1 x_1 = m_2 x - m_2 x_1$

(or) $m_1 x_1 + m_2 x_1 = m_2 x$

(or) $(m_1 + m_2) x_1 = m_2 x$

$$\therefore x_1 = \frac{m_2 x}{m_1 + m_2} \quad (4)$$

From equation (1), we can also write $x_1 = x - x_2$ (5)

Similarly, by substituting equation (5) in equation (2), we get

$$m_1 (x - x_2) = m_2 x_2$$

(or) $m_1 x - m_1 x_2 = m_2 x_2$

(or) $m_1 x = m_1 x_2 + m_2 x_2$

(or) $m_1 x = (m_1 + m_2) x_2$

$$\therefore x_2 = \frac{m_1 x}{m_1 + m_2} \quad (6)$$

Moment of inertia

The moment of inertia (I) of a diatomic molecule with respect to an axis passing through centre of mass of the system shall be written as

$$I = m_1 x_1^2 + m_2 x_2^2 \quad (7)$$

Substituting equation (4) and equation (6) in equation (7), we get

$$I = m_1 \left[\frac{m_2 x}{(m_1 + m_2)} \right]^2 + m_2 \left[\frac{m_1 x}{(m_1 + m_2)} \right]^2$$

(or) $I = \dfrac{x^2}{(m_1 + m_2)^2} \left[m_1 . m_2^2 + m_2 . m_1^2 \right]$

(or) $I = \dfrac{x^2 (m_1 . m_2)}{(m_1 + m_2)^2} \left[m_1 + m_2 \right]$

(or) $I = \dfrac{(m_1 . m_2)}{(m_1 + m_2)} x^2$ (8)

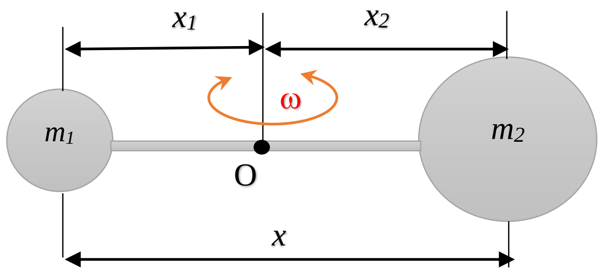
Since $\mu = \frac{(m_1.m_2)}{(m_1 + m_2)}$ is called the reduced mass of the system, we can write equation (8) as

$I = \mu x^2$ (9)

Equation (9) represents the moment of inertia of a diatomic molecule.

1.5.7. Rotational Energy state of a rigid diatomic molecule

Let us consider a rigid diatomic molecule having two atoms of masses m_1 and m_2 connected by a weightless rod of length x. This rigid diatomic molecule rotates with an angular velocity ω with respect to an axis through the centre of mass O and is perpendicular to the connecting rod as shown in figure.



We know that the kinetic energy of rotating diatomic molecule is $K.E. = \frac{1}{2} I \omega^2$ (1)

We know that the angular momentum of a rotating body is $L = I\omega$ (or) $\omega = \frac{L}{I}$ (2)

Substituting equation (2) in (1), we get $K.E. = \frac{1}{2} I \frac{L^2}{I^2}$

(or) $K.E. = \frac{L^2}{2I}$ (3)

We know that the moment of inertia of a rotating diatomic molecule is $I = \mu x^2$ (4)

Substituting equation (4) in (3), we get,

Kinetic energy $K.E. = \frac{L^2}{2\mu x^2}$ (5)

Equation (5) represents the classical equation for kinetic energy of a rigid diatomic molecule, in which the energy levels are continuous for all possible values of 'L'.

But according to quantum mechanics, we know that the energy values are discrete.

Based on quantum theory, the angular momentum L shall be written as $L = \sqrt{J(J+1)}.\hbar$ (6)

Where J is the total angular momentum quantum number and its values are 0, 1, 2, 3,… so on.

Substituting equation (6) in equation (5), we get $E_J = \frac{J(J+1).\hbar^2}{2\mu x^2}$ (7)

This equation (7) represents the rotational kinetic energy of a rigid diatomic molecule, quantum mechanically.

Special cases

When J = 0, equation (7) becomes, $E_0 = 0$

When J = 1, equation (7) becomes, $E_1 = \frac{2.\hbar^2}{2\mu x^2} \ (or) E_1 = \frac{\hbar^2}{\mu x^2}$ (8)

When J = 2, equation (7) becomes, $E_2 = \frac{2(3).\hbar^2}{2\mu x^2} \ (or) E_1 = \frac{3\hbar^2}{\mu x^2}$ (9)

From eqn. (8) and (9), $E_2 = 3 E_1$

When J = 3, equation (7) becomes, $E_2 = \frac{3(4).\hbar^2}{2\mu x^2} \ (or) E_1 = \frac{6\hbar^2}{\mu x^2}$ (10)

From eqn. (8) and (10), $E_3 = 6 E_1$

Therefore, In general, $E_J = \frac{J(J+1)}{2} E_1$

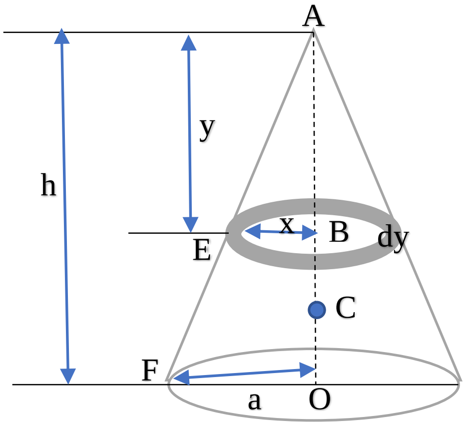
From these results, we can confirm that rotational kinetic energy of rigid diatomic molecule is quantized and discrete.

1.5.8. Problems in Centre of Mass

Problem 1: centre of mass of a solid cone

Let us consider a solid circular cone of base radius *a* and height *h*. let ρ be the density of the material of the cone. If the solid cone is homogeneous, then its mass

$$m = \frac{1}{3}\pi a^2 h.\rho$$



The centre of mass lies on the axis of symmetry AO. The cone is connected to made up of large number of circular discs, each of thickness *dy*.

Let us consider one such elementary disc of radius *x* at a distance *y* from the vertex *A* of the solid cone. The mass of this elementary disc is

$dm = \rho\,(\pi x^2)\,dy$ (1)

From the figure, $\frac{x}{a} = \frac{y}{h}$

(or) $x = \frac{a}{h}y$

$$\therefore dm = \rho.\pi.\left(\frac{a}{h}y\right)^2.dy \quad (2)$$

Now from equation (1), we have for the distance of centre of mass on the axis of symmetry *AO* as measured from the vertex *A* as

$$Y_{CM} = \frac{1}{M}\int y.dm \quad (3)$$

Substituting equation (2), in (3), we get

$$Y_{CM} = \frac{1}{M}\int y.\rho.\pi.\left(\frac{a}{h}y\right)^2.dy$$

(or) $$Y_{CM} = \frac{\rho.\pi.a^2}{M.h^2}\int y^3.dy \quad (4)$$

Where the limits of *y* is taken from *y* = 0 to *y* = *h* to cover the entire solid cone filled with such elementary discs. It gives

$$Y_{CM} = \frac{\rho.\pi.a^2}{M.h^2}\left[\frac{y^4}{4}\right]_0^h$$

$$Y_{CM} = \frac{\rho.\pi.a^2}{M.h^2}\times\frac{h^4}{4} = \frac{\rho.\pi.a^2.h^2}{M}$$

But *M* = Total mass of the solid cone = $\frac{1}{3}\pi a^2 h.\rho$

Hence, $$Y_{CM} = \frac{\rho.\pi.a^2.h^2.3}{4.\pi.a^2.h.\rho}$$

The CM of cone from its vertex Y_{CM} is written as R_{CM}

$$R_{CM} = \frac{3}{4}h$$

Thus, CM of a solid cone is at a distance of ¾ *h* from vertex of the cone along the axis.

Problem 2: Moment of inertia of a thin uniform rod

Position 1

About an axis through its centre of mass and perpendicular to its length

Let PQ be a thin uniform rod of length *l* & mass *M*. The rod is free to rotate abut an axes XX' perpendicular to tis length and passing through the centre of mass 'O'.

Mass per unit length of the rod (linear density) $m = \frac{M}{l}$ (1)

Consider a small element *dx* at a distance *x* from 'O'

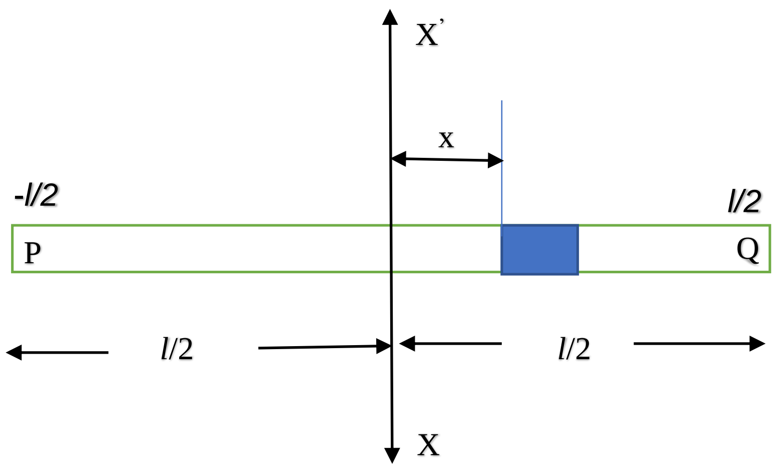
Mass of the element (*M*) = *m. dx*

Moment of inertia of this element about XX' = mass x (distance)^2

$= m\, dx \,.\, x^2$ (2)

The rod consists of number of such elements of length *dx* . Hence the moment of inertia *I* of the rod about XX' is obtained by integrating equation (1) between x = -*l/2* to x = *l/2.*

$$\therefore I = \int_{-\frac{l}{2}}^{\frac{l}{2}} m.x^2 dx \quad (3)$$



$$I = m\left[\frac{x^3}{3}\right]_{-\frac{l}{2}}^{\frac{l}{2}}$$

$$I = m\left[\frac{\frac{l^3}{8} + \frac{l^3}{8}}{3}\right]$$

(or) $I = m\left[\frac{l^3}{12}\right]$

(or) $I = ml\left[\frac{l^2}{12}\right]$

(or) $I = \left[\frac{Ml^2}{12}\right]$ (4)

Where M = *m l*

About an axis passing through one end of the rod and perpendicular to its length

Let PQ be a thin uniform rod of length *l* an mass *M*. O is its centre. As the rod is uniform, its centre and centre of gravity coincides. XX' is an axis passing through O and perpendicular to the length of the rod.

Moment of inertia of the rod about XX' = $\left[\frac{Ml^2}{12}\right]$ (5)

Let AA' be an axis passing through one end P an perpendicular to the length of the rod. Let I be the moment of inertia of the rod about this axis AA'

By parallel axis theorem,

$I_{AA'} = I_{xx'} + Mx^2$ (6)

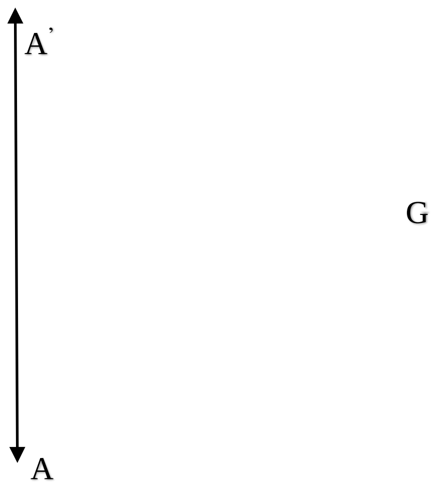
Here the distance x = *l/2*, hence, substituting this and equation (5) in (6), we get

$$I_{AA'} = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2$$

(or) $I_{AA'} = \frac{Ml^2}{12} + \frac{Ml^2}{4}$

(or) $I_{AA'} = \frac{4Ml^2}{12}$

(or) $I_{AA'} = \frac{Ml^2}{3}$



Problem 3: moment of inertia of a circular ring

Let us find the moment of inertia of a circular ring with rotating axis at various points.

Position1

Rotating axis is passing through the centre of mass (ring centre) and perpendicular to the ring plane

Let us consider a circular ring with radius 'R' and mass 'M' rotating about an axis passing through the centre of ring 'O' as shown in figure. Let us consider a elemental portion of the ring (*dl*) at the circumference of the ring (*L*) and the mass of the elemental ring is *'dm'*

Therefore, the moment of inertia of the elemental ring is given by

$$dI = (dm)R^2 \quad (1)$$

Here, the mass of an elemental portion '*dm*' of the ring is

Mass (*dm*) = Length mass density (μ) X Length of the elemental portion of ring (*dl*)

$$\therefore dm = \mu dl \quad (2)$$

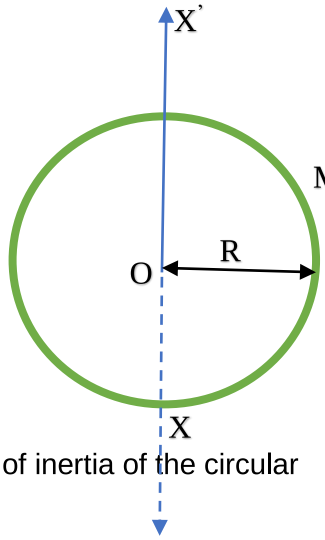
We know, the length mass density of the ring is

$$\mu = \frac{\text{Mass(M)}}{\text{Circumferencial length(L)}} = \frac{M}{2\pi R} \quad (3)$$

Where *R* is the radius of the ring

Substituting equation (3) in (2), we get

$$dm = \frac{M}{2\pi R} dl \qquad (4)$$



Substituting equation (4) in (1), we get

$$dI = \frac{M}{2\pi R} dl \times R^2$$

(or) $$dI = \frac{MR}{2\pi} dl \qquad (5)$$

Since the circular ring is a continuous body, we can get the moment of inertia of the circular ring by integrating equation (5) within the limits of 0 to 2πR.

$$\therefore \int dI = \int_0^{2\pi R} \frac{MR}{2\pi} dl$$

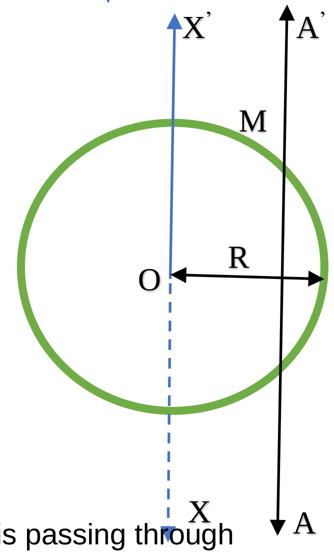
(or) $$\int dI = \frac{MR}{2\pi} \int_0^{2\pi R} dl$$

(or) $$I = \frac{MR}{2\pi} [l]_0^{2\pi R}$$

(or) $$I = \frac{MR}{2\pi} [2\pi R]$$

(or) $$I = MR^2$$

Therefore, the moment of inertia of the circular ring when the rotating axis passing through centre of mass is $I = M R^2$. (6)



Position 2

Rotating axis at the edge of the ring and perpendicular to the ring plane

Let AA' be the rotation axis at the edge of the ring which is perpendicular to the ring plane as shown in figure. Here, we can see that XX' axis is passing through the centre of mass of the ring is parallel to AA' axis.

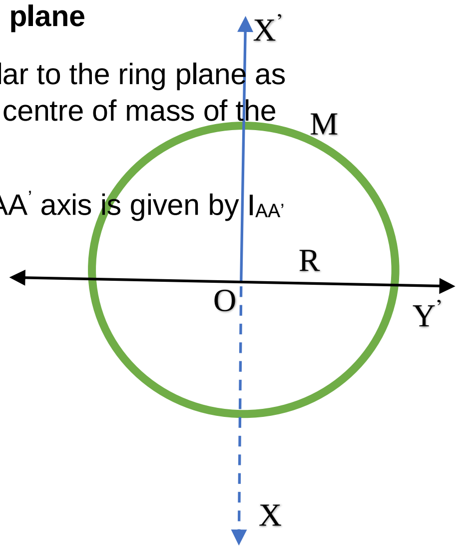
Based on parallel axis theorem, the moment of inertia with respect to AA' axis is given by $I_{AA'} = I_{XX'} + MR^2$ (7)

Using equation (6), $I_{XX'} = MR^2$ (8)

Substituting equation (8) in (7), we get

$I_{AA'} = MR^2 + MR^2$

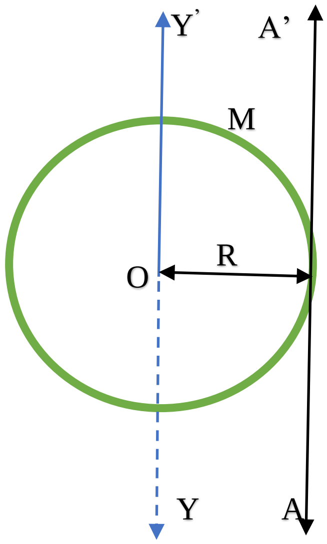
Therefore, $I_{AA'} = 2MR^2$ (9)



Position 3

Rotating axis is passing through the diameter of the ring

Let YY' be the rotating axis passing through the diameter of the ring, which is perpendicular to XX' as shown in figure



Based on perpendicular axis theorem, we can write

$I_{XX'} = I_{YY'} + I_{ZZ'}$ (10)

Here for circular disc, $I_{ZZ'} = I_{YY'}$

Therefore, equation (10) can be written as

$I_{XX'} = I_{YY'} + I_{YY'}$

(or) $I_{XX'} = 2I_{YY'}$

(or) $I_{YY'} = \frac{I_{XX'}}{2}$ (11)

Using equation (6), we can write $I_{XX'} = MR^2$, hence equation (11) becomes,

$$I_{YY'} = \frac{MR^2}{2} \quad (12)$$

Equation (12) represents the moment of inertia when the rotating axis is passing through the diameter of the ring.

Position 4

Rotating axis at the edge of the ring and parallel to ring plane

Let AA' be the rotating axis at the edge of the ring and parallel to the ring plane.

Let YY' be the axis that passes through the diameter of the ring, which is parallel to AA' as shown in figure

Based on parallel axis theorem, the moment of inertia with respect to AA' axis is given by
$I_{AA'} = I_{YY'} + MR^2$ (13)

Using equation (12), we can write $I_{YY'} = \frac{MR^2}{2}$ (14)

Substituting equation (14) in (13), we get

$$I_{AA'} = \frac{MR^2}{2} + MR^2$$

(or) $I_{AA'} = \frac{3}{2}MR^2$ (15)

Equation (15) is the moment of inertia when the rotating axis is at edge of the ring and parallel to ring plane.

Problem 4: Moment of inertia of a circular disc

Position 1

Rotating axis is passing through the centre of mass and perpendicular to the disc plane

Let us consider a circular disc with radius *R* rotating about an axis passing through the centre of the disc *O*. Let the mass of the disc *M* be uniformly distributed all over the surface area of the disc.

The disc shall be assumed to contain infinitesimally small rings. Let us consider one such ring of the mass *dm* and thickness *dr*, which is located at a distance *r* from the centre of the disc *O* as shown in figure.

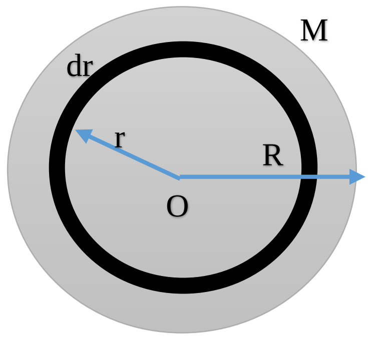
Therefore, The moment of inertia of a small ring is given by

$dI = (dm)\, r^2$ (1)

Here, the mass of small ring *(dm)* with radius *r* is given by

Mass (dm) = Surface density *x* circumference of the ring *x* Thickness of the ring

$\therefore dm = (\sigma).(2\pi r).(dr)$ (2)



We know that, the surface mass density for the disc

$$\sigma = \frac{Mass(M)}{Area(A)} = \frac{M}{\pi R^2}$$ (3)

Substituting equation (3) in equation (2), we get

$$dm = \frac{M}{\pi R^2} 2\pi r.dr$$

$$\therefore dm = \frac{2M}{R^2} r.dr$$ (4)

Substituting equation (4) in equation (1), we get

$$dI = \frac{2M}{R^2} r.dr.r^2$$

(or) $dI = \frac{2M}{R^2} r^3.dr$ (5)

Since the circular disc is a continuous body, we can get the moment of inertia of the entire disc by integrating equation (5) within the limits 0 to R,

$$\therefore \int dI = \int_0^R \frac{2M}{R^2} r^3 dr$$

(or) $$I = \frac{2M}{R^2} \int_0^R r^3 dr$$

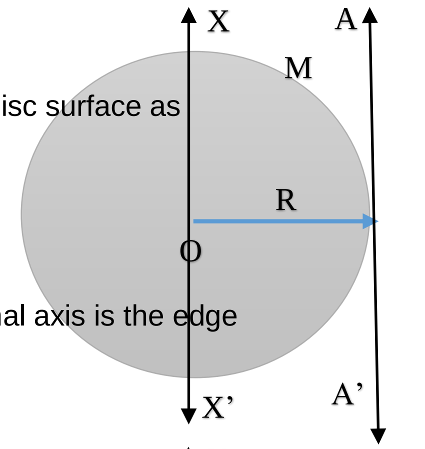
(or) $$I = \frac{2M}{R^2} \left[\frac{r^4}{4} \right]_0^R$$

(or) $I = \frac{2M}{R^2} . \frac{R^4}{4}$

The moment of inertia of a circular disc $I = ½ MR^2$ (6)

Equation (6) represents the moment of inertia of a circular disc when the rotation axis is passing through the centre of mass.

Position 2



Rotating axis at the edge of the disc and perpendicular to the disc plane

Let XX' and AA' axis are parallel and both the axis are perpendicular to disc surface as shown in figure.

Based on parallel axis theorem

$I_{AA''} = I_{XX'} + MR^2$ (7)

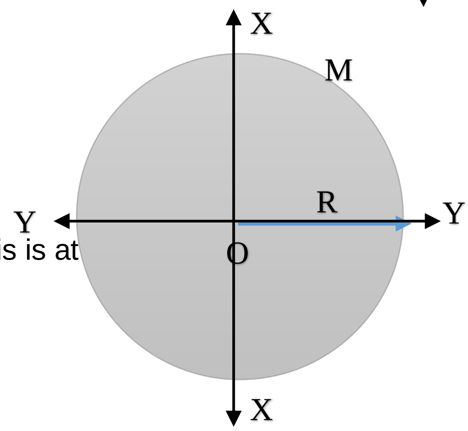
Here, $I_{AA'}$ is the moment of inertia of the circular disc fir which the rotational axis is the edge of the disc.

Using equation (6), we can write $I_{XX'} = ½ MR^2$ (8)

Substituting equation (8) in (7), we get

$I_{AA'} = ½ MR^2 + MR^2$

$I_{AA'} = \frac{3}{2} MR^2$ (9)



Equation (9) represents the moment of inertia, when the rotational axis is at disc.

Position 3

Rotating axis is passing through the diameter of the disc

Let YY' be the rotating axis passing through the diameter of the disc, which is perpendicular to XX' as shown in figure

Based on perpendicular axis theorem, we can write

$I_{XX'} = I_{YY'} + I_{ZZ'}$ (10)

Here for circular disc, $I_{ZZ'} = I_{YY'}$

Therefore, equation (10) can be written as

$I_{XX'} = I_{YY'} + I_{YY'}$

(or) $I_{XX'} = 2I_{YY'}$

(or) $I_{YY'} = \frac{I_{XX'}}{2}$ (11)

Using equation (6), we can write $I_{XX'} = ½ MR^2$, hence equation (11) becomes,

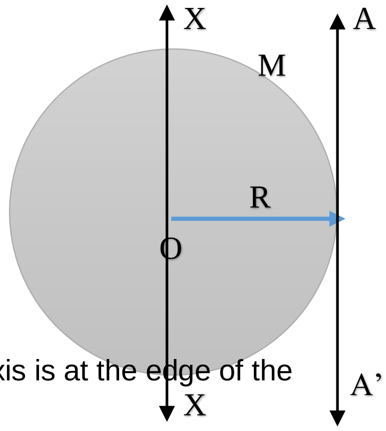
$$I_{YY'} = \frac{1}{4}MR^2 \qquad (12)$$

Equation (12) represents the moment of inertia, when the rotational axis is passing through the diameter of the disc.

Position 4

Rotating axis at the edge of disc and parallel to disc plane

Let YY’ and AA’ axes are parallel to each other and also parallel to disc surface as shown in figure.



Based on parallel axis theorem, $I_{AA''} = I_{XX'} + MR^2$ (13)

Substituting (13) in (12), we get

$$I_{AA'} = MR^2 + \frac{1}{4}MR^2$$

$$\therefore I_{AA'} = \frac{5}{4}MR^2 \qquad (14)$$

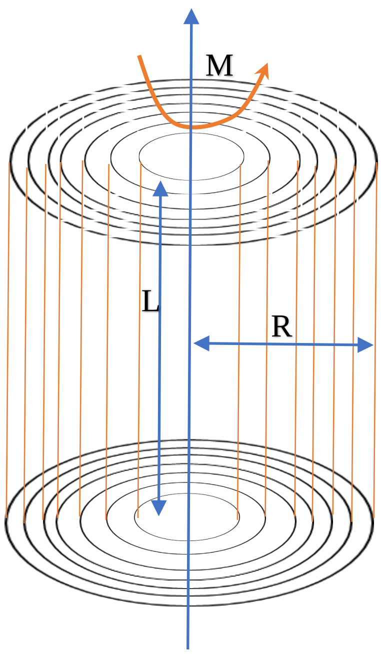
Equation (14) represents the moment of inertia, when the rotational axis is at the edge of the disc an parallel to the plane.

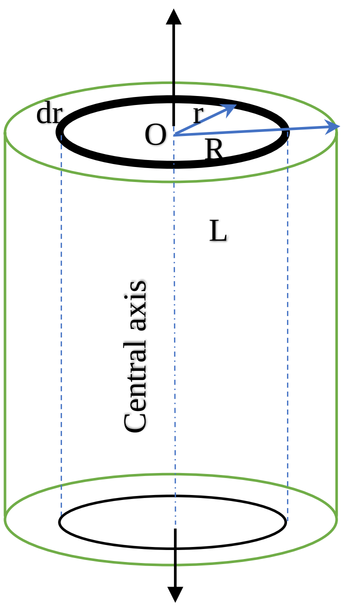
1.21 Moment of inertia of a solid cylinder

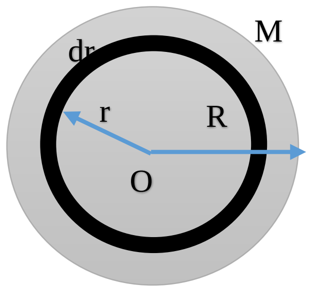
Let us consider a solid cylinder of mass *M* , length *L* an radius *R* which contains infinitesimally thin cylinders as shown in figure.

Here the mass is uniformly distributed all over the solid cylinder, rotating about the central axis. Let us consider one such thin cylinder having mass *dm* thickness *dr* and length *L* located at a distance *r* from central axis of the cylinder as shown in figure

The top view of solid cylinder (of radius *R*) and thin cylinder (of radius *r*) are shown in figure







The moment of inertia of the thin cylinder is given by

$dI = (dm)\, r^2$ (1)

Here, the mass of the thin cylinder *(dm)* is

Mass (dm) = Volume density x Area x Length

(or) dm = Volume density x circumference x thickness x length

Therefore, $dm = \rho \, . \, 2\pi r. \, dr \, . \, L$ (2)

We know that, the surface mass density for the disc

$$\sigma = \frac{Mass(M)}{Area(A)} = \frac{M}{\pi R^2 L}$$ (3)

Substituting equation (3) in equation (2), we get

$$dm = \frac{M}{\pi R^2 L} 2\pi r.dr.L$$

$$\therefore dm = \frac{2M}{R^2} r.dr$$ (4)

Substituting equation (4) in equation (1), we get

$$dI = \frac{2M}{R^2} r.dr.r^2$$

(or) $$dI = \frac{2M}{R^2} r^3.dr$$ (5)

Since the circular disc is a continuous body, we can get the moment of inertia of the entire disc by integrating equation (5) within the limits 0 to R,

$$\therefore \int dI = \int_0^R \frac{2M}{R^2} r^3 dr$$

(or) $$I = \frac{2M}{R^2} \int_0^R r^3 dr$$

(or) $$I = \frac{2M}{R^2} \left[\frac{r^4}{4} \right]_0^R$$

(or) $$I = \frac{2M}{R^2} . \frac{R^4}{4}$$

The moment of inertia of a circular disc $I = ½\, MR^2$ (6)

Equation (6) represents the moment of inertia of a solid cylinder with respect to central axis.

1.22. Moment of inertia of a hollow cylinder

Let us consider a hollow cylinder of inner radius R_1 an outer radius R_2 with length L which is rotating about the central axis as shown in figure.

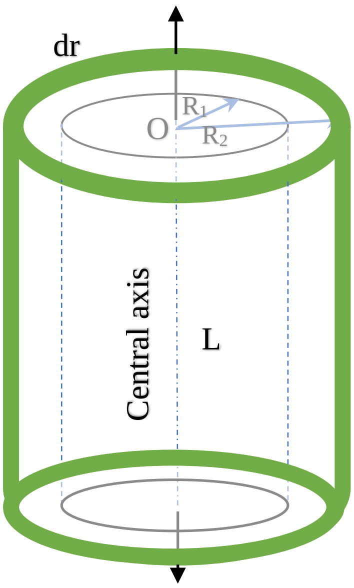
Here the mass is uniformly distributed all over the solid cylinder, rotating about the central axis. Let us consider one such thin cylinder having mass *dm* thickness *dr* and length *L* located at a distance *r* from central axis of the cylinder as shown in figure

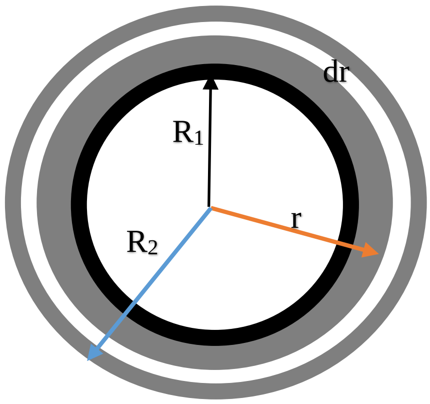
The top view of solid cylinder (of radius *R*) and thin cylinder (of radius *r*) are shown in figure.

Therefore, the moment of inertia of this thin layer of cylinder is given by $dI = (dm)\, r^2$ (1)

Here, the mass of a thin cylinder (*dm*) is

Mass (*dm*) = Volume density x Area x Length





(or) dm = Volume density x circumference x thickness x length

Therefore, $dm = \rho \,.\, 2\pi r \,.\, dr \,.\, L$ (2)

We know that, the surface mass density for the disc

$$\sigma = \frac{Mass(M)}{Area(A)} = \frac{M}{\pi(R_2^{\,2} - R_1^2)L} \qquad (3)$$

Substituting equation (3) in equation (2), we get

$$dm = \frac{M}{\pi(R_2^2 - R_1^2)L} 2\pi r.dr.L$$

$$\therefore dm = \frac{2M}{(R_2^2 - R_1^2)} r.dr \qquad (4)$$

Substituting equation (4) in equation (1), we get

$$dI = \frac{2M}{(R_2^2 - R_1^2)} r.dr.r^2$$

(or) $$dI = \frac{2M}{(R_2^2 - R_1^2)} r^3.dr \quad (5)$$

Since the circular disc is a continuous body, we can get the moment of inertia of the entire disc by integrating equation (5) within the limits 0 to R,

$$\therefore \int dI = \int_{R_1}^{R_2} \frac{2M}{(R_2^2 - R_1^2)} r^3 dr$$

(or) $$I = \frac{2M}{(R_2^2 - R_1^2)} \int_{R_1}^{R_2} r^3 dr$$

(or) $$I = \frac{2M}{(R_2^2 - R_1^2)} \left[\frac{r^4}{4} \right]_{R_1}^{R_2}$$

(or) $$I = \frac{2M}{(R_2^2 - R_1^2)} . \frac{(R_2^4 - R_1^4)}{4}$$

(or) $$I = \frac{1}{2} . \frac{M(R_2^2 - R_1^2).(R_2^2 + R_1^2)}{(R_2^2 - R_1^2)}$$

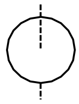
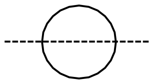
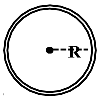
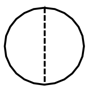
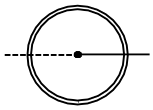
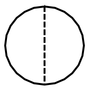
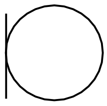
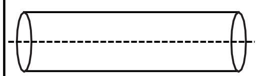
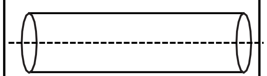
The moment of inertia of a circular disc $I = ½ M (R_2^2 + R_1^2)$ (6)

Equation (6) represents the moment of inertia of a hollow cylinder with respect to central axis.

If the wall of hollow cylinder is very thin, then $R_1 \cong R_2$, which is equal to *R*,

Then Moment of inertia of thin wall hollow cylinder is $I = ½ M (R^2 + R^2)$

$(or)\ I = MR^2$

Body	**Location Axis**	**Figure**	**MI**
A thin uniform rod of length l	Passing through the C.G and $\perp r$ to the length		$\frac{M\,l^2}{12}$
A thin uniform rod of length l	Passing through one end and $\perp r$ to the length		$\frac{M\,l^2}{3}$
Circular thin ring of radius R	Passing through the centre and $\perp r$ to the plane		MR^2
Circular thin ring of radius R	About any diameter		$\frac{MR^2}{2}$
Circular thin ring of radius R	About a tangent in the plane of ring		$\frac{3}{2}MR^2$
Circular thin circular disc of radius R	About an axis through it centre		$\frac{MR^2}{4}$
Circular thin circular disc of radius R	About any diameter		$\frac{MR^2}{4}$
Solid sphere of radius R	About any diameter		$\frac{2}{5}MR^2$
Solid sphere of radius R	About any tangent		$\frac{7}{5}MR^2$
Solid cylinder of radius R and length l	Passing through CM and $\perp r$ to the length		$M\left[\frac{l^2}{12}+\frac{R^2}{4}\right]$
Solid cylinder of radius R and length l	About symmetry axis		$\frac{MR^2}{2}$